Thickness and Information in Dynamic Matching Markets

Mohammad Akbarpour
Stanford Graduate School of Business

Joint work with
Shengwu Li & Shayan Oveis Gharan

Matching Markets

Kidney Exchange

School Choice

Labor Markets

Dating

NRMP

Foster Care

Static Matching Markets

Theory:

[Gale-Shapley, 1962], [Shapley-Shubik, 1971], [Shapley-Scarf, 1971], [Kelso-Crawford, 1982], [Roth, 1982, 1984], [Immorlica-Mahdian, 2005], [Hatfield-Milgrom, 2005], [Che-Kojima, 2007], [Ostrovsky, 2008], [Kojima-Pathak, 2009], [Kojima-Manea, 2009], [Budish, 2011], [Budish-Che-Kojima-Milgrom, 2013], [Kojima-Pathak-Roth, 2013], [Hatfield-Kominers-Nichifor-Ostrovsky-Westkamp, 2013],[Echenique-Lee-Shum-Yenmez, 2013], …

School Choice:

[Abdulkadiroglu-Pathak-Roth, 2005, 2009], [Abdulkadiroglu-Pathak-Roth, 2005, 2006], [Pathak-Sonmez, 2013], [Abdulkadiroglu-Angrist-Dynarski-Kane-Pathak, 2011], …

Kidney Exchange:

[Roth-Sonmez-Unver, 2003, 2005, 2007], [Abraham-Blum-Sandholm, 2007], [Unver, 2010], [Ashlagi-Roth, 2013], [Ashlagi-Gamarnik-Rees-Roth, 2012], …

Other Applications

[Peranson-Roth, 1999], [Jolls-Posner-Roth, 2001], [Sonmez-Switzer, 2013], [Che-Koh, 2014], [Pycia-Unver, 2014], …

Given a set of agents

Find matching algorithms with some desirable properties:

Stability

Efficiency

Strategy-proofness

…

The Static Question

Gale-Shapley (1962):

"The match is made, and all is done!"

The Taming of the Shrew, William Shakespeare

Which agents to match?

Dynamic Matching Markets

Kidney Exchange

School Choice

Labor Markets

Dating

NRMP

Foster Care

The composition of options is endogenously determined by the matching algorithm

A New Question

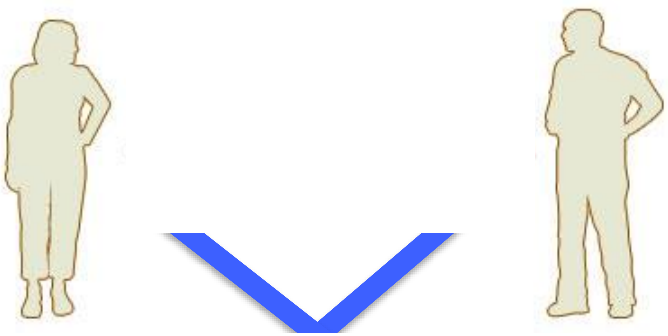
Which agents to match?

(Widely studied)

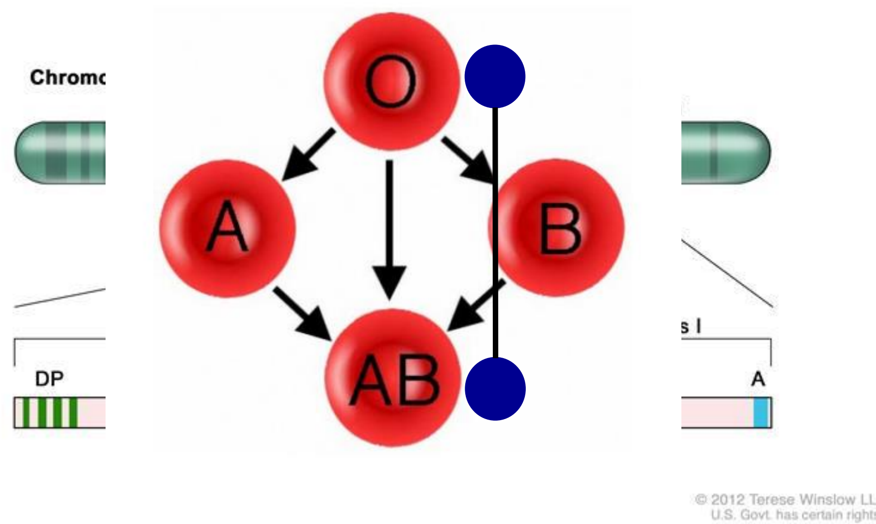
When to match agents?

This Talk

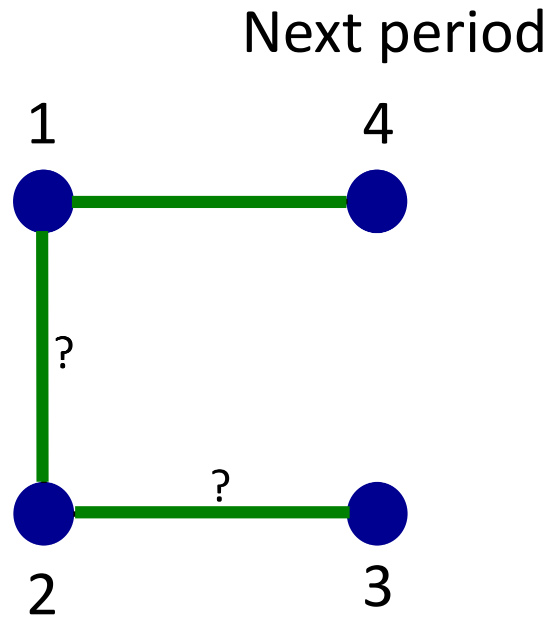
Motivating Example: Kidney Exchange



Biological compatibility:

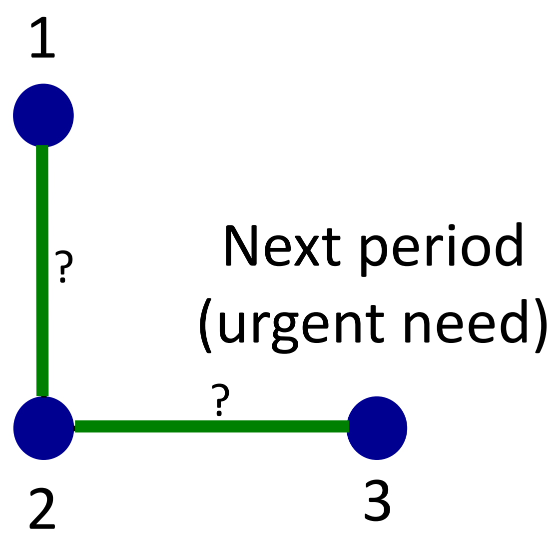


Value of Waiting: More Information



1- Future trade network (*i.e.* new matching opportunities)

Value of Waiting: More Information



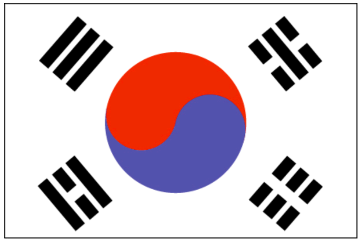
1- Future trade network (*i.e.* new matching opportunities)

2- Agents' urgency of needs

Questions about Timing

- How significant is the (option) value of waiting?
- What is the optimal waiting time?
- What kind of information is valuable?
- Do agents have incentive to misreport something?

Timing in Kidney Exchange

ALLIANCE FOR PAIRED DONATION	UNOS DONATE LIFE — UNITED NETWORK FOR ORGAN SHARING		
Daily	Weekly	Monthly	Quarterly

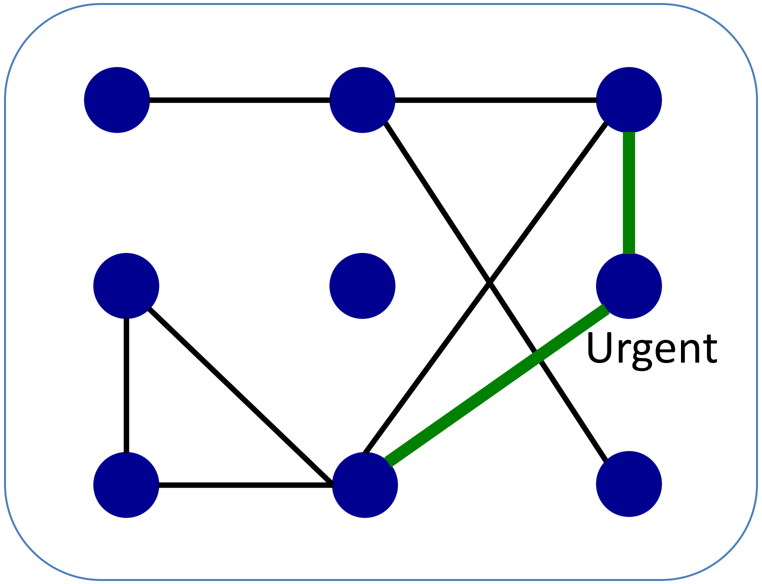
This Paper: A New Model

- Agents arrive and depart continuously over time
- Explicit modeling of the matching network
- A central planner observes the network, and agents who are about to depart, and continuously matches agents
- The goal is to maximize social welfare

This Paper: Main Findings

1- **Value of waiting** can be very large

- Waiting thickens the trade network (*i.e.* provides *liquidity*)



- So, we can react to urgent cases with high probability

This Paper: Main Findings

2- **Information** of agents' urgency of needs is highly valuable

- The planner can be patient with respect to those who are not in urgent need, thus maintain market thickness.

3- **Incentive-Compatibility**: When urgency information is private, we design a dynamic mechanism (without transfers) to extract it.

Related Literature (Static)

Theory:

[Gale-Shapley, 1962], [Shapley-Shubik, 1971], [Shapley-Scarf, 1971], [Kelso-Crawford, 1982], [Roth, 1982, 1984], [Immorlica-Mahdian, 2005], [Hatfield-Milgrom, 2005], [Che-Kojima, 2007], [Ostrovsky, 2008], [Kojima-Pathak, 2009], [Kojima-Manea, 2009], [Budish, 2011], [Budish-Che-Kojima-Milgrom, 2013], [Kojima-Pathak-Roth, 2013], [Hatfield-Kominers-Nichifor-Ostrovsky-Westkamp, 2013],[Echenique-Lee-Shum-Yenmez, 2013], …

School Choice:

[Abdulkadiroglu-Pathak-Roth, 2005, 2009], [Abdulkadiroglu-Pathak-Roth, 2005, 2006], [Pathak-Sonmez, 2013], [Abdulkadiroglu-Angrist-Dynarski-Kane-Pathak, 2011], …

Kidney Exchange:

[Roth-Sonmez-Unver, 2003, 2005, 2007], [Abraham-Blum-Sandholm, 2007], [Unver, 2010], [Ashlagi-Roth, 2013], [Ashlagi-Gamarnik-Rees-Roth, 2012], …

Other Applications

[Peranson-Roth, 1999], [Jolls-Posner-Roth, 2001], [Sonmez-Switzer, 2013], [Che-Koh, 2014], [Pycia-Unver, 2014], …

Related Literature (Dynamic)

- See the related work section of the paper

Outline

- Setup
  - A Model of Dynamic Matching
  - Designing Matching Algorithms
- Main Results
  - Value of Waiting
  - Value of Information & Mechanism Design
- Extensions
  - Welfare under Discounting and Optimal Waiting Time
  - Increasing Trade Frequency
- Concluding discussions

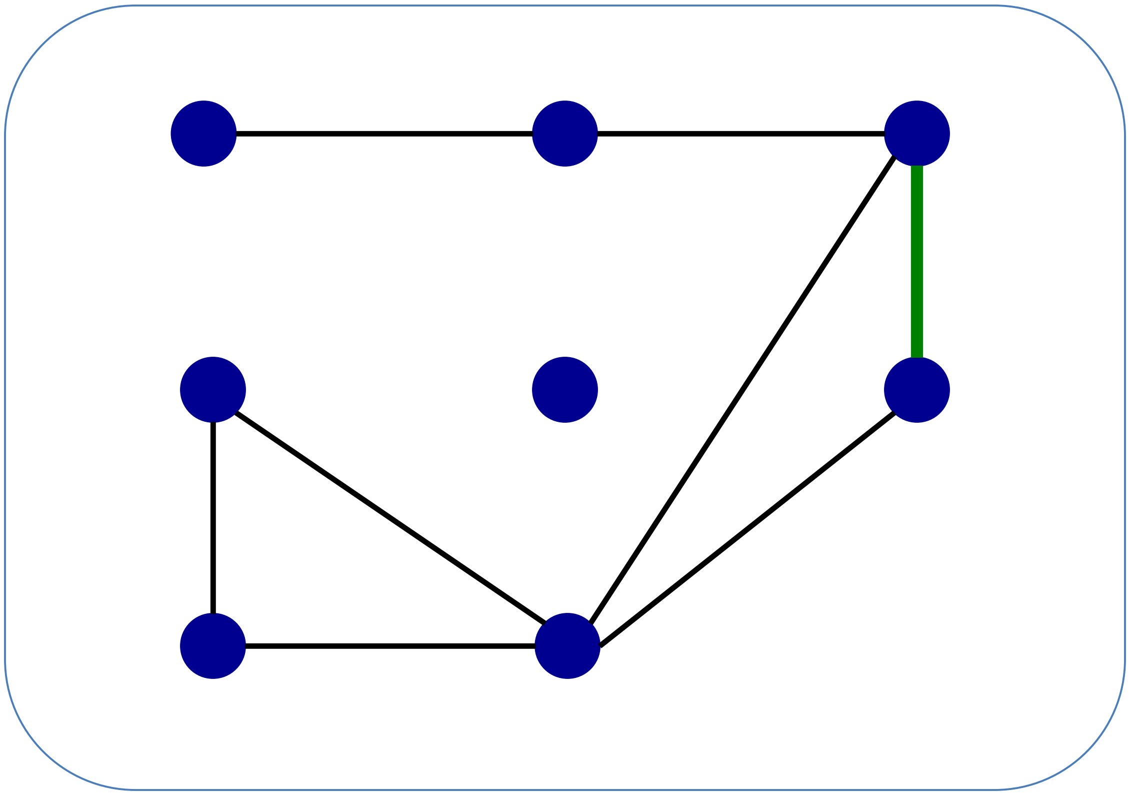
Model

- Agents arrive continuously with rate m

- There is an acceptable transaction between any two agents with i.i.d probability p

- Each agent gets *critical* independently with rate 1
- Agents depart when
  - get matched
  - get critical and perish

Model: Illustration



Model: Two Key Parameters

- Agents arrive continuously with rate m
- There is an acceptable transaction between any two agents with i.i.d probability p
- Each agent gets *critical* indep. with rate 1

d º $m \times p$ (from now on: $\mathbf{p = d / m}$)

Proxy for average degree (or *network sparsity*)

Matching Algorithm

The Planner observes:

- Set of agents in the pool (*nodes)*
- The set of acceptable transactions (*edges*)

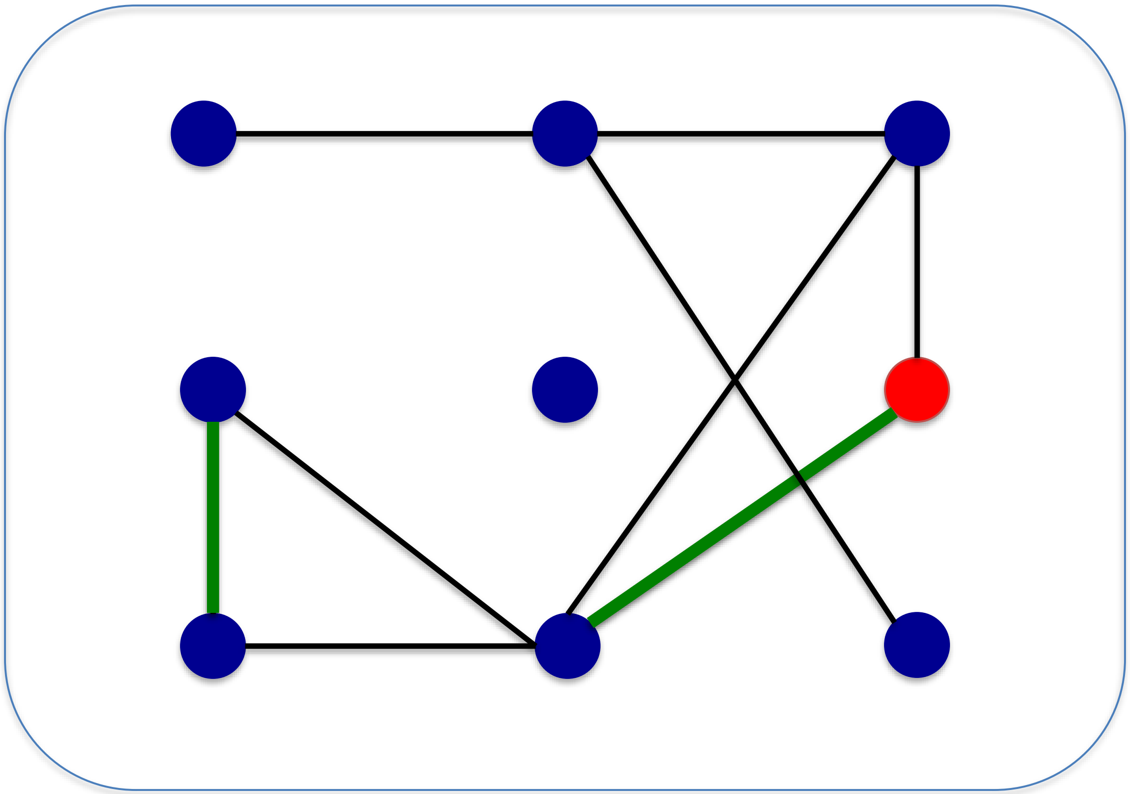
G(t): *Trade Possibilities Network*

- The Planner observes critical agents. (relax later)

- A Dynamic Matching Algorithm: **Γ: G(t) → M** (red "c" above G)

A set of disjoint edges (possibly empty)

Matching: Illustration



Goal

Suppose waiting cost is zero. (relax later)

Minimize expected fraction of perished agents.

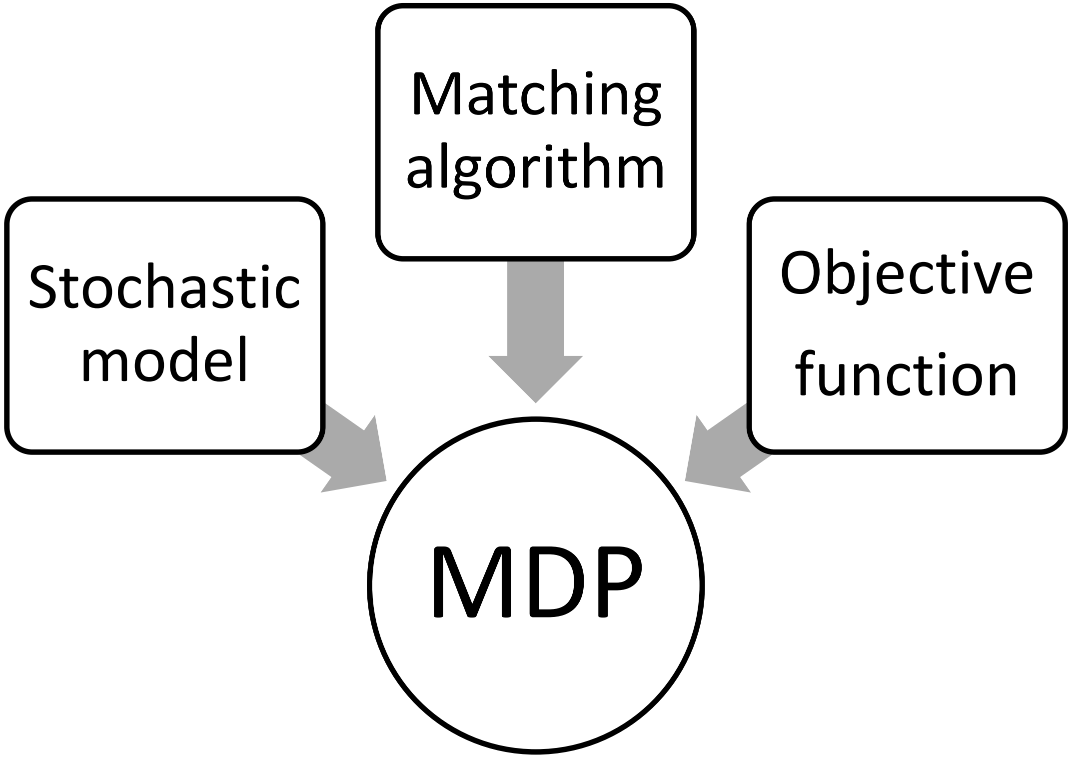
Agents who leave unmatched

Definition. For an algorithm ALG, target time T,

$$\text{Loss}(\text{ALG},T) := \frac{E[\#\text{ of perished agents}]}{m \times T}$$

(Expected) # of agents who arrive by time T

A Markov Decision Problem



# of networks on on ***n*** nodes ≈ $2^{O(n^2)}$

Computationally Complex

Designing Matching Algorithms: Towards Optimum

Simple Local Matching Algorithms

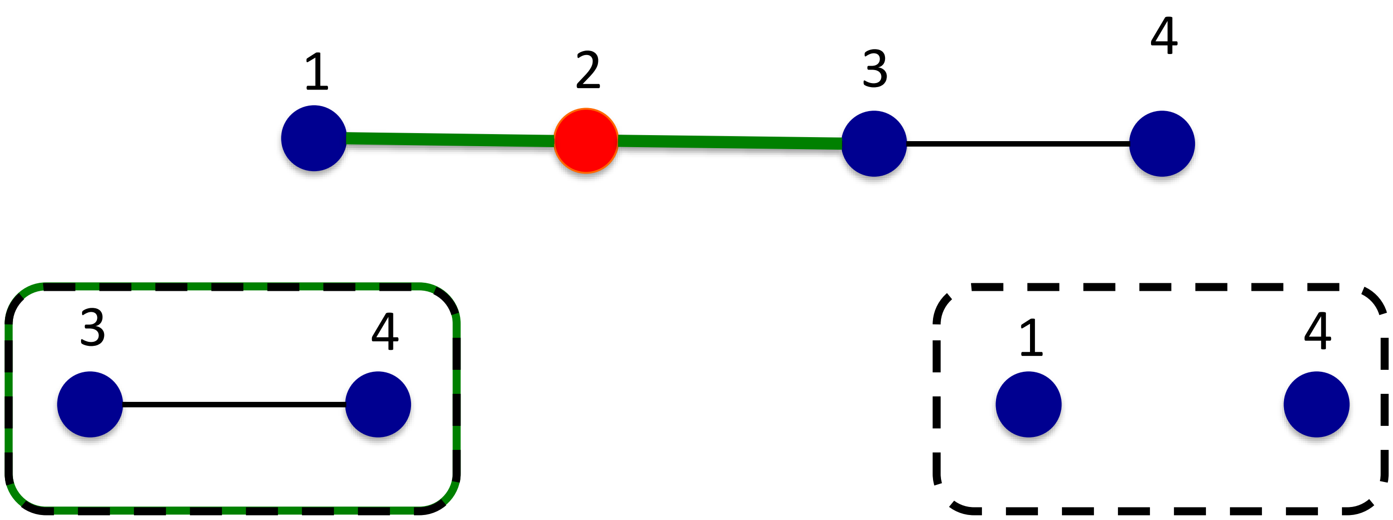
1- *Greedy* Algorithm: Match agents upon their arrival to a random neighbor (if any).

2- *Patient* Algorithm: Match agents when they get critical to a random neighbor (if any).

Patient: Smart in 'When', Naïve in 'Who'

Patient chooses the optimal *time* to match an agent.

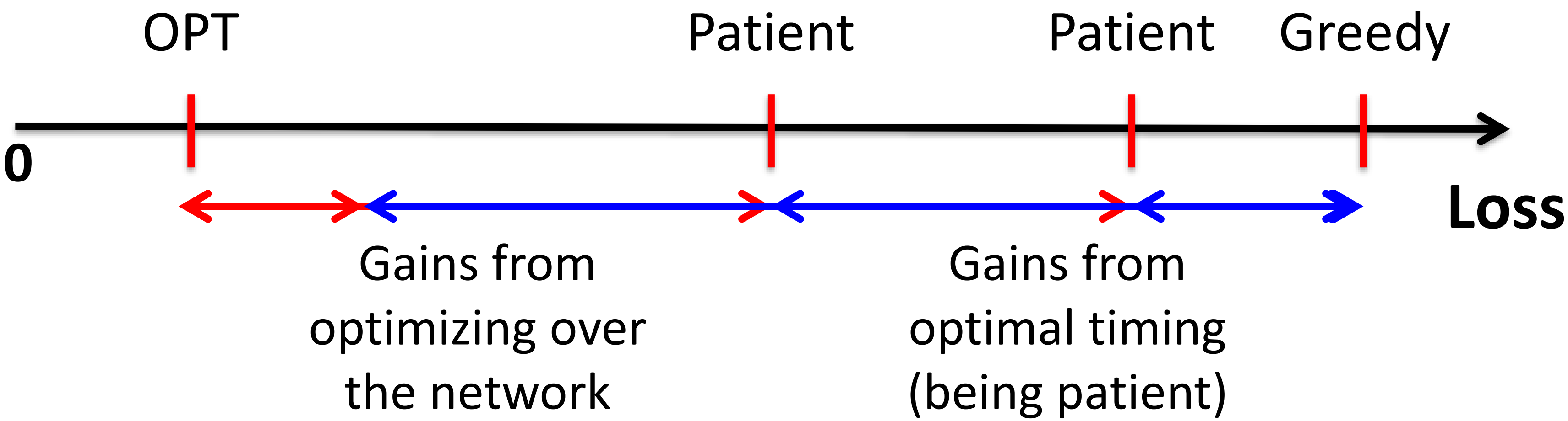
But it is naïve in optimizing over the network structure.



Outline

- **Setup**
  - **A Model of Dynamic Matching**
  - **Designing Matching Algorithms**
- Main Results
  - Value of Waiting
  - Value of Information & Mechanism Design
- Extensions
  - Welfare under Discounting and Optimal Waiting Time
  - Increasing Trade Frequency
- Concluding discussions

Comparing Algorithms



For this talk: (all are carefully discussed in the paper)

- *Steady State*
- *Relatively large values of* ***m***
- ***d*** *> 2*

Value of Waiting

Theorem: In steady state, for large values of **m**,

1: Loss(Greedy) ≥ $\mathbf{1/(2d+1)}$

2: Loss(Patient) ≤ $\mathbf{e^{-d/2}/2}$

As a result,

$$\text{Loss(Patient)} \leq (\mathbf{d + 1/2}) \cdot \mathbf{e^{-d/2}} \cdot \text{Loss(Greedy)}$$

For d=8,

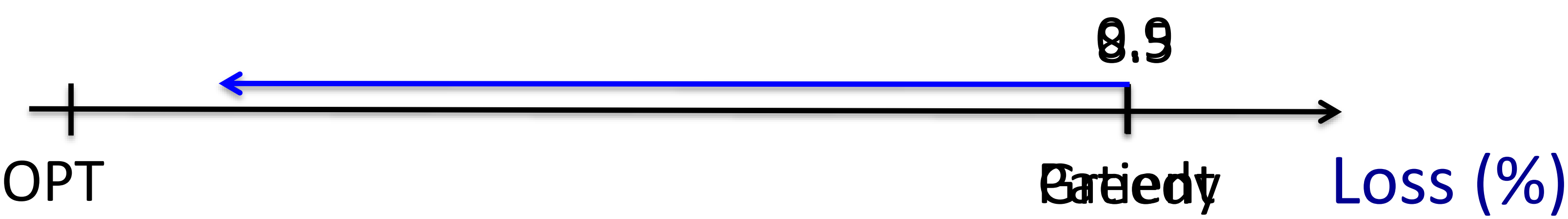
$$\text{Loss(Patient)} \leq \mathbf{0.17} \cdot \text{Loss(Greedy)}$$

Timing vs. Optimization

Theorem: In steady state, for large values of **m**,

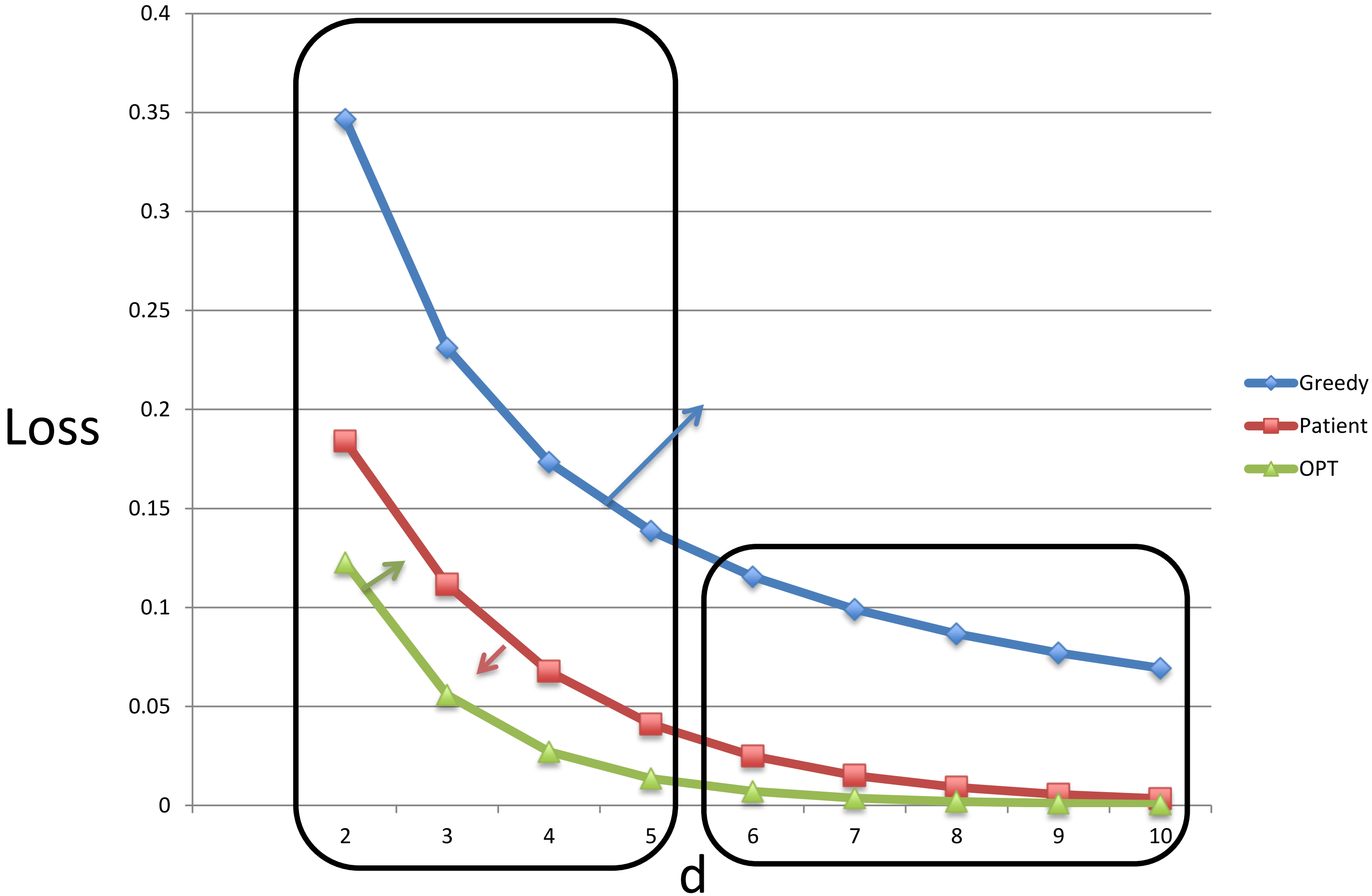
$$\mathbf{e^{-d}/(d+1)} \leq \text{Loss(OPT)} \leq \text{Loss(Patient)} \leq \mathbf{e^{-d/2}/2}$$

d = 8,



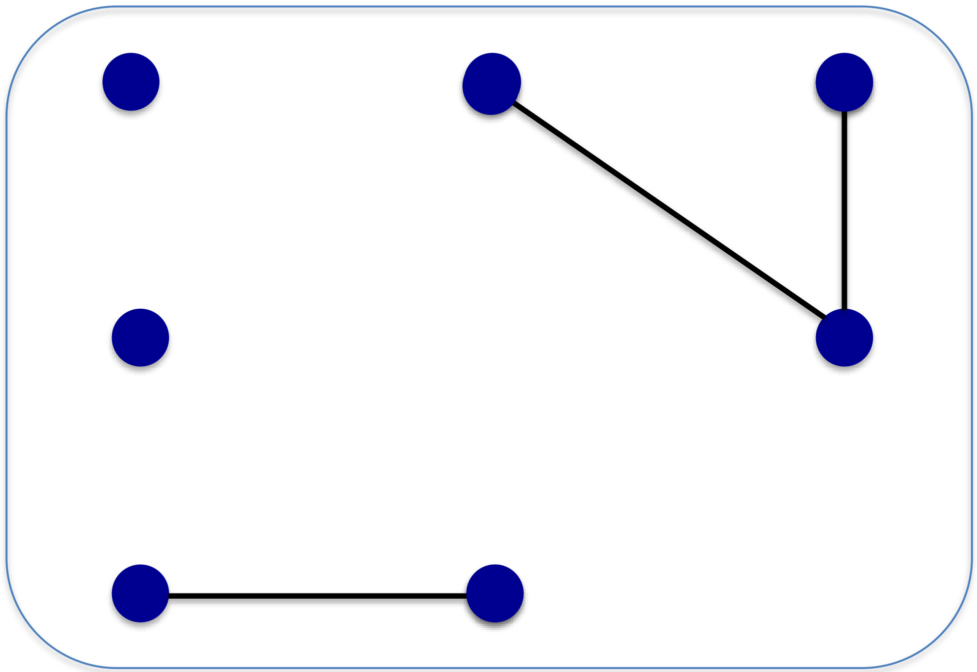
Most of the gain is achieved by merely being patient

Greedy vs. Patient vs. OPT



Proof Ideas

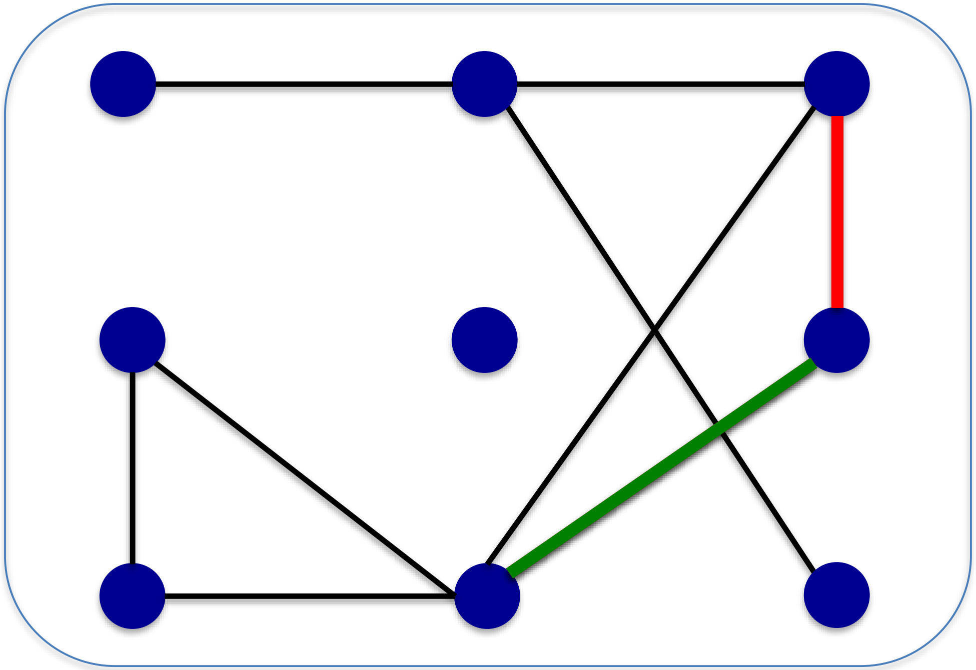
Greedy: Composition of Market



The graph of agents (pool) is always an *empty graph*

Perishing rate = criticality rate · 1 = *pool size*

Patient: Composition of Market



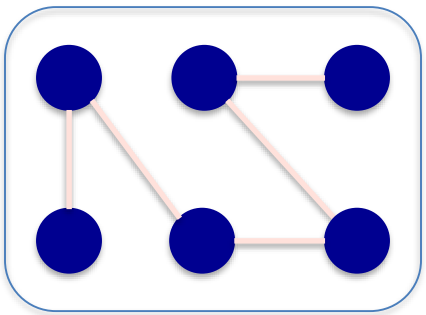
The pool is always *Erdős–Rényi* with parameter **d/m**

Perishing rate = *pool size* · $(1 – d/m)^{pool\ size\ -\ 1}$ P(# matches = 0)

Bounding Losses

Suppose $Z_t \approx \mathbf{E(Z_t)}$ (pool size is highly concentrated)

Patient



Perishing rate

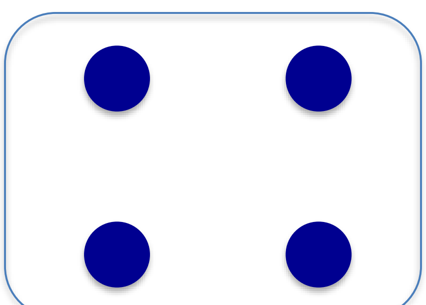
$$\text{Loss} \approx E(Z_t)\cdot(1-d/m)^{E(Z_t)-1} / m$$

Arrival rate

$E(Z_t) \geq \mathbf{m/2}$

$$\text{Loss} \leq \mathbf{e^{-d/2} / 2}$$

Greedy



Perishing rate

$$\text{Loss} \approx E(Z_t) \cdot 1 / m$$

Arrival rate

$E(Z_t) \geq \mathbf{m/(2d+1)}$

$$\text{Loss} \geq \mathbf{1/(2d+1)}$$

Key Findings, So Far

1- Patience can be highly valuable:

$$\text{Loss(Patient)} \leq \mathbf{(d+1/2)} \cdot \mathbf{e^{-d/2}} \cdot \text{Loss(Greedy)}$$

2- Most of the gain is achieved by being patient.

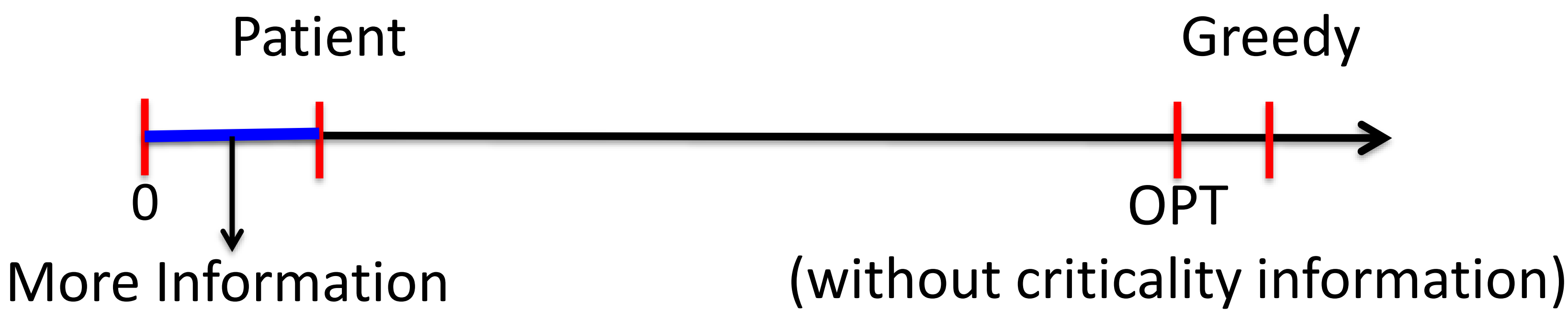
*"How poor are they that have not **patience**!*
*What wound did ever heal but by **degrees**?"*

Othello (II, iii, p376)
William Shakespeare

Outline

- **Setup**
  - **A Model of Dynamic Matching**
  - **Designing Matching Algorithms**
- **Main Results**
  - **Value of Waiting**
  - Value of Information & Mechanism Design
- Extensions
  - Welfare under Discounting and Optimal Waiting Time
  - Increasing Trade Frequency
- Concluding discussions

Value of Information



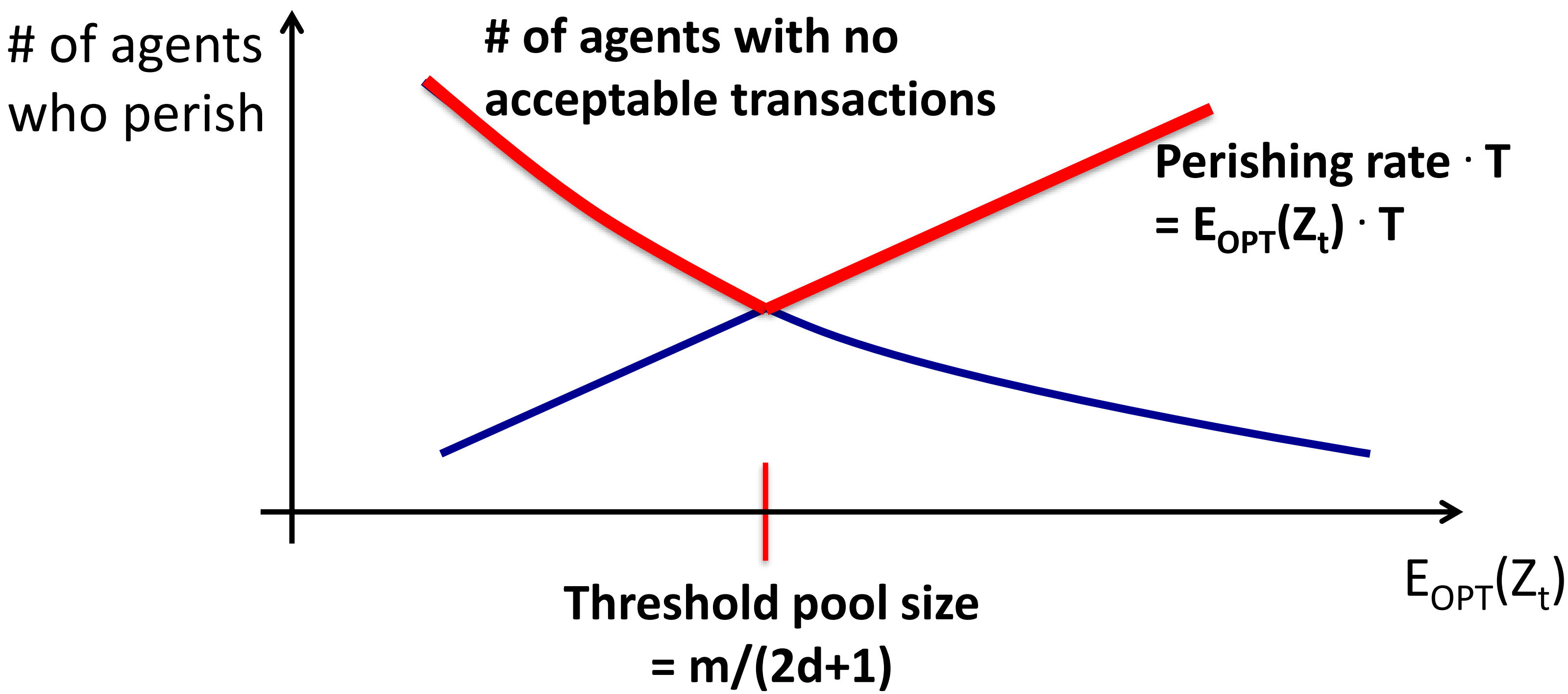
Theorem: Without criticality information,

1/(2d+1) ≤ Loss(OPT) ≤ Loss(Greedy) ≤ **ln(2)/d**

Criticality information and waiting are complements.

OPT Performance

$E_{OPT}(Z_t)$: expected value of pool size under OPT



$$\text{Loss(OPT)} \geq \frac{m/(2d+1)}{m} = 1/(2d+1)$$ **QED.**

Information Structure and Utilities

Information Structure:

- **Agents** observe:
    - When they are critical
    - Underlying model parameters (m, d, 1)
    - Do **not** observe the exact trade network

Discount rate   Time spent in pool

$$u(a) := \begin{cases} e^{-r \cdot s(a)} & \text{If a is matched} \\ 0 & \text{Otherwise} \end{cases}$$

- **Planner** observes:
    - The exact trade network
    - Does **not** observe when agents are critical

A Dynamic Mechanism

Patient-Mechanism:

- Ask for agents’ departure times.

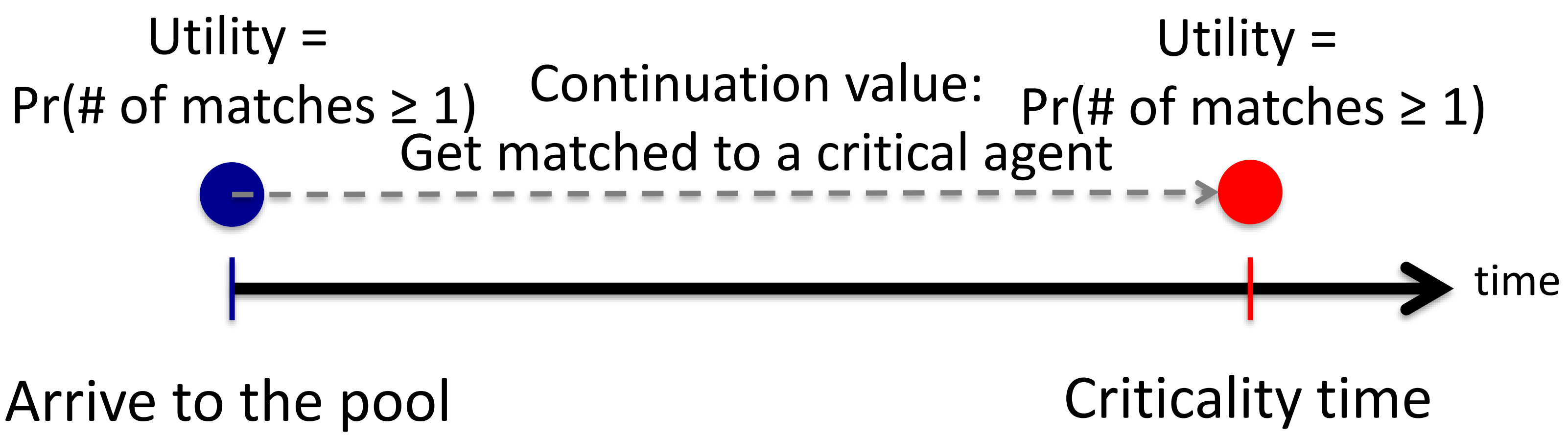
- When an agent announces getting critical, match her to a random neighbor.

- If she has no neighbors, never match her again.

Incentive Compatibility

Theorem. There exists a $r_1 > 0$ such that for any $r \leq r_1$, the truthful strategy profile is an ε-Nash equilibrium for Patient-Mechanism, where $\varepsilon \rightarrow 0$ as $m \rightarrow \infty$.

Continuation Value



Problem: By being in the pool, agents learn about its distribution and update their beliefs.

Solution: Show that agents' posterior beliefs cannot go outside of the "concentration interval".

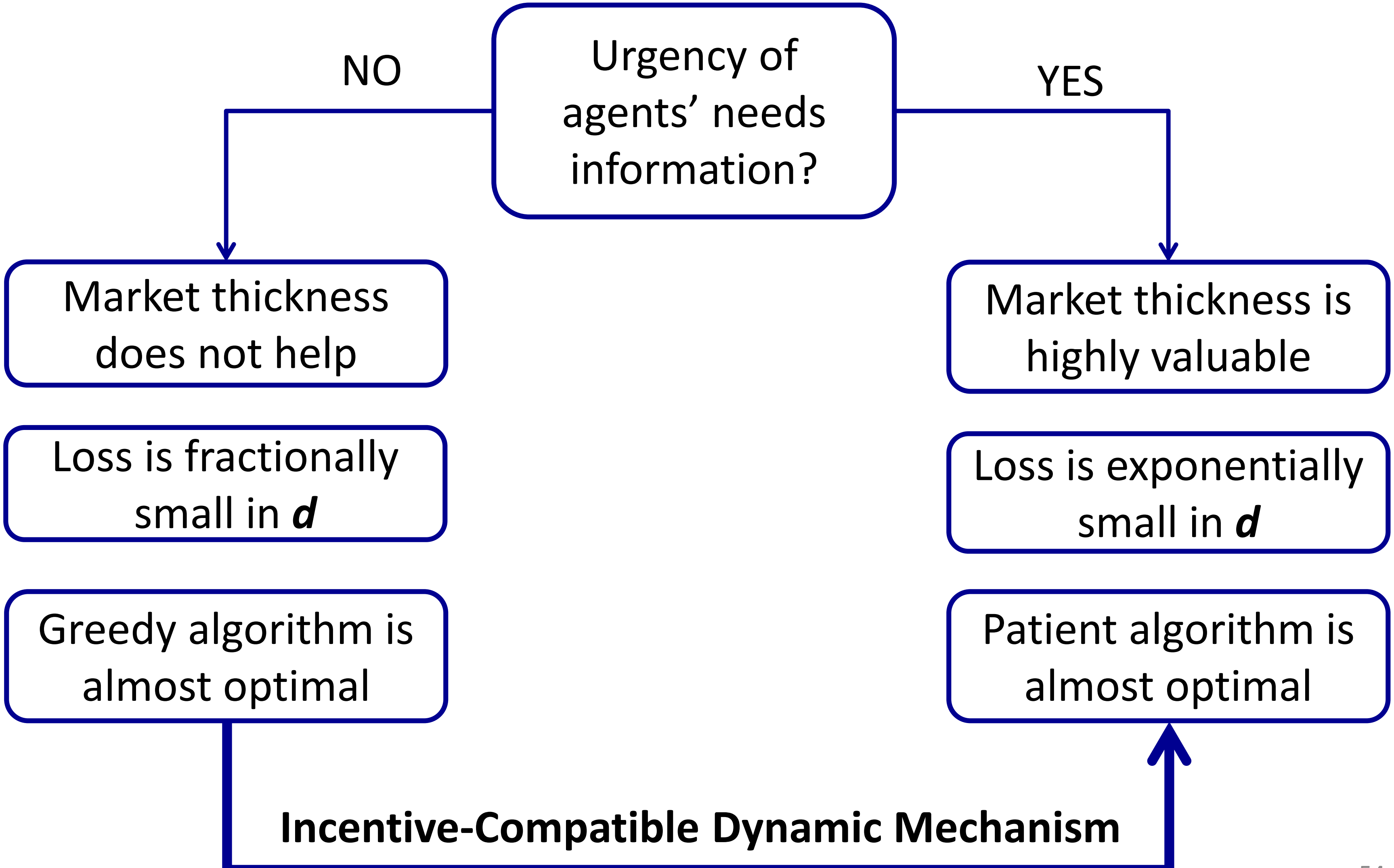
Hard to Commit: A New Punishment

Can we commit to kick agents out if they lie?

Different Punishment

If an agent lied, keep her in the pool, but assign the lowest priority to her when a critical agent has multiple neighbors.

Summary of Findings



Reasons to Be Greedy

- Waiting cost is high

- No information about agents' urgency of needs

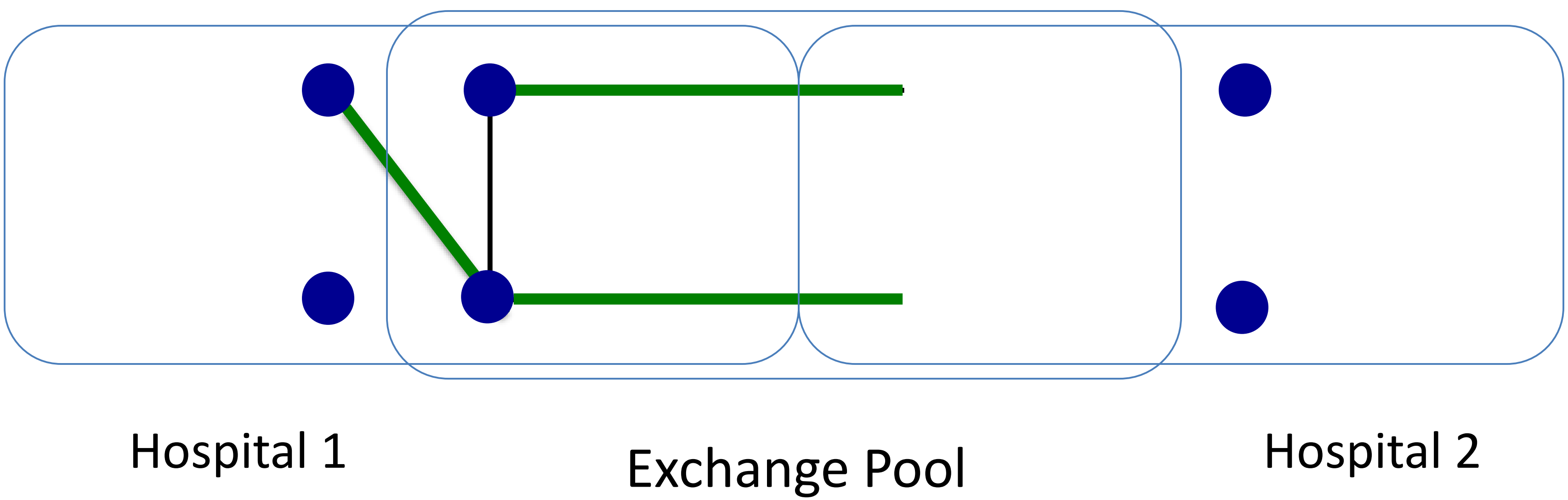
- If **p** is very small or very large, Greedy and Patient's performances are close. (extreme cases: **p=0** or **p=1**)

Key Findings

- When composition of market is a function of matching policy, market thickness (liquidity) is a key concern
- The information of urgency of agents' needs is very valuable, and it can be extracted with simple mechanisms without transfers
- The optimal waiting time is decreasing in waiting cost, arrival rate of agents, and match probabilities

A Lesson for Kidney Exchange

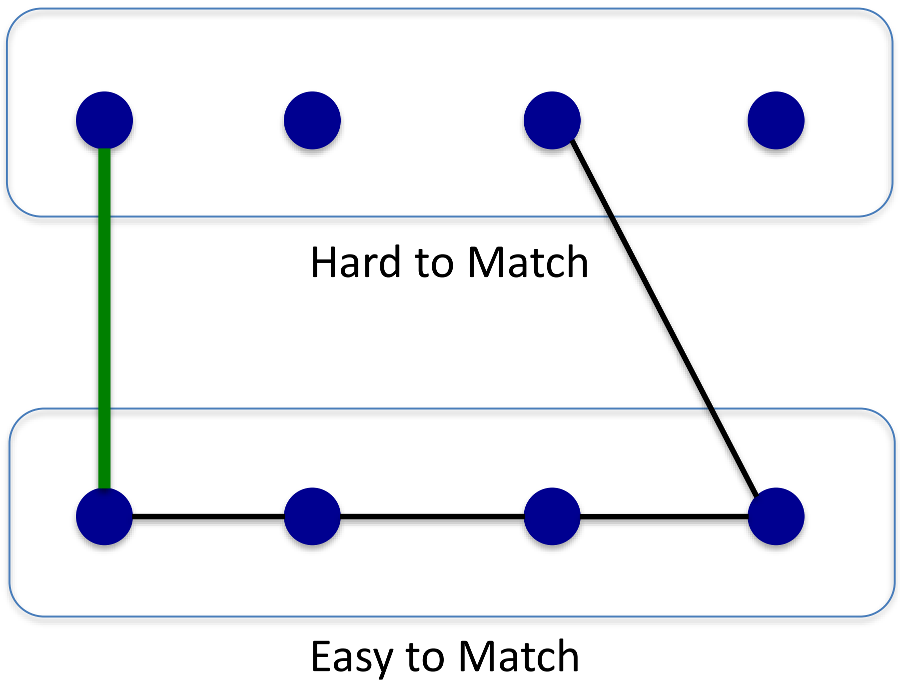
Multi-hospital issues: “Greedy” behavior of hospitals is very costly.



Assumption: *Ex ante* Homogeneous

In a multiple type model, tie breaking matters more.

[*Akbarpour, Nikzad, Rees, Roth, 2015 (working paper)*]



Much Remains to Be Done

- Dynamics are important in many markets:

- We showed:
  - Timing can be a first-order concern
  - ***Dynamic networked markets*** can be analytically studied by exploiting tools from algorithm design and stochastic processes

- ***Much*** work remains to be done:
  - Decentralized markets and prices
  - Platform competition
  - Dynamic stability

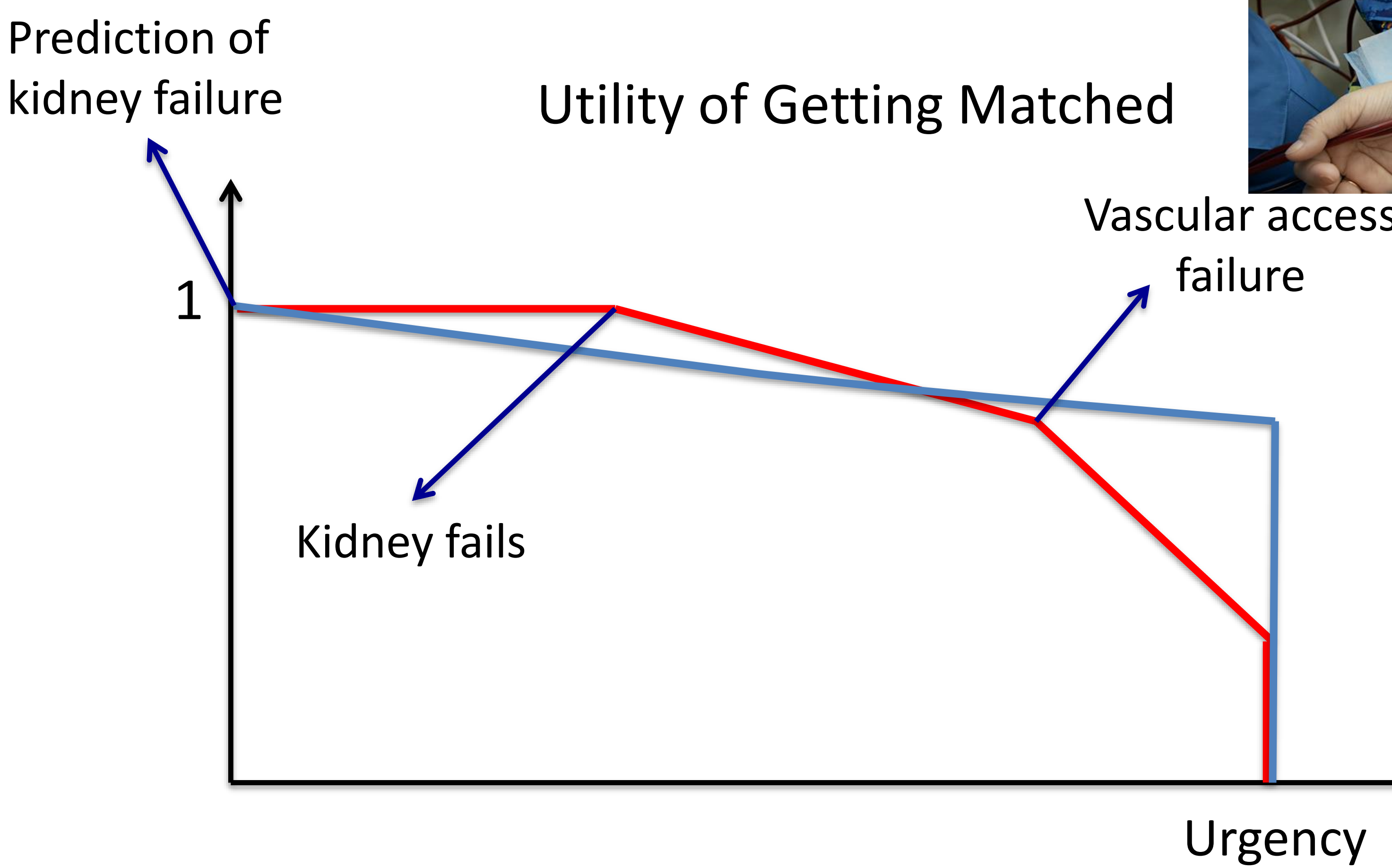
Last Policy Implication

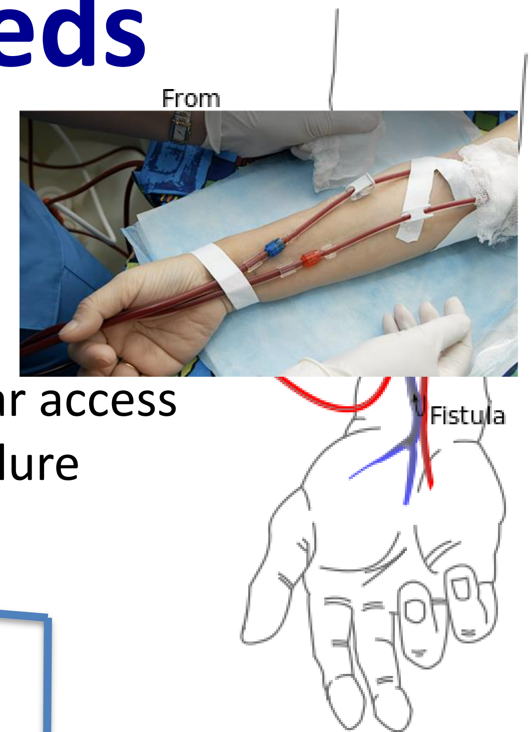
Even the optimal algorithm cannot match all patients…

So, drink more water to prevent kidney failure!

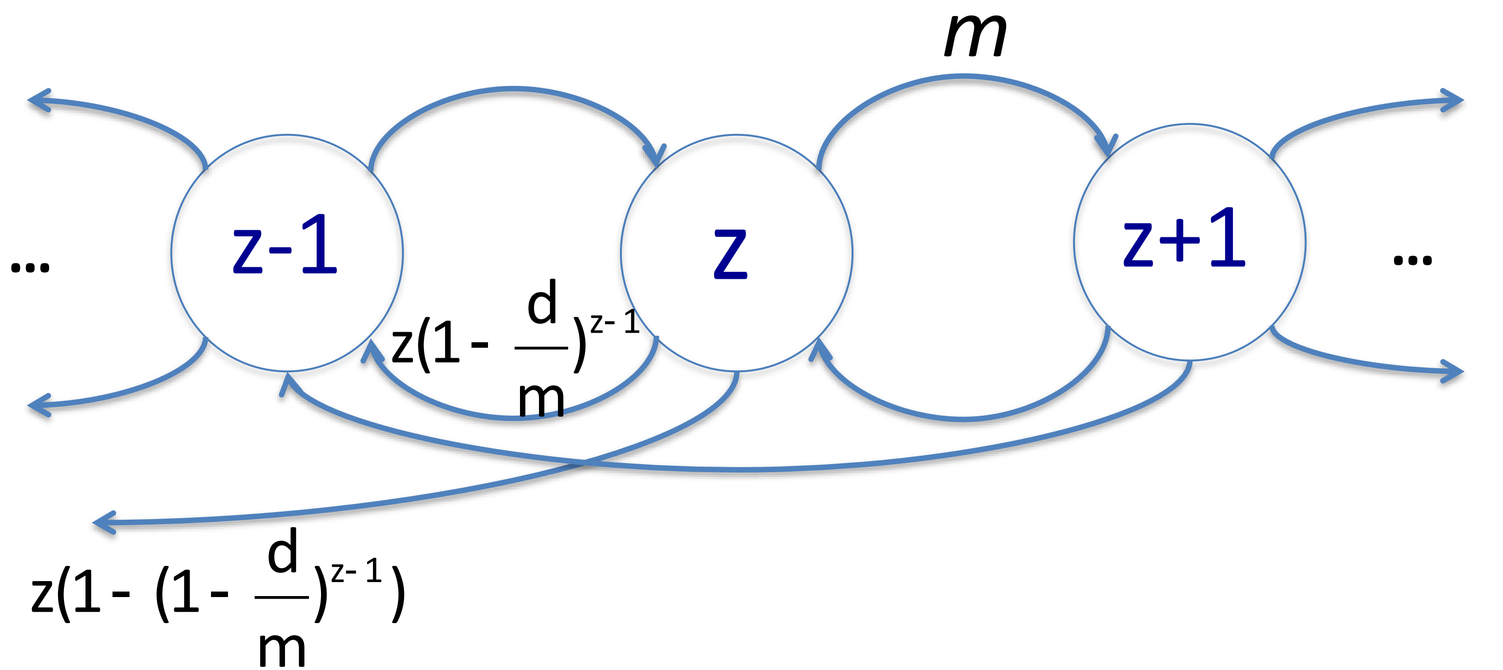
Thank you!

Utility and Urgency of Needs



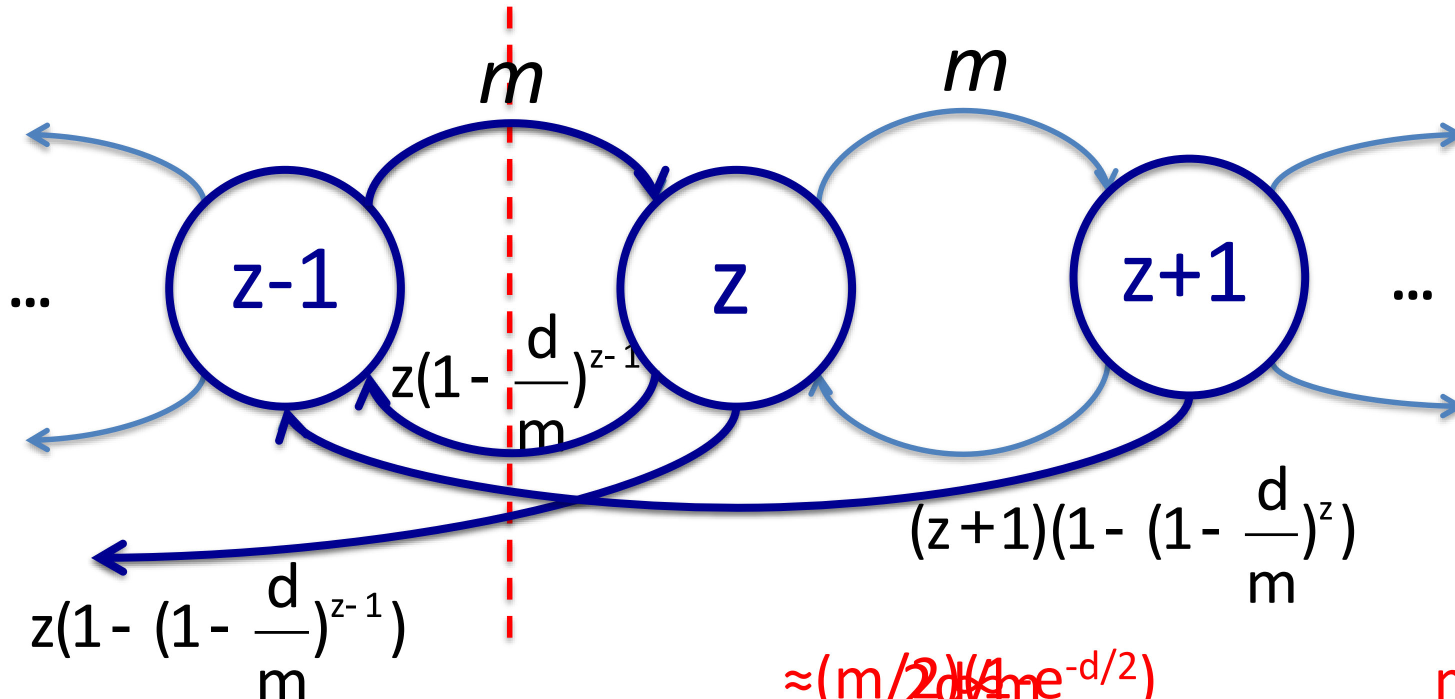


Patient Pool Size Markov Chain



No closed form expression for stationary distribution !

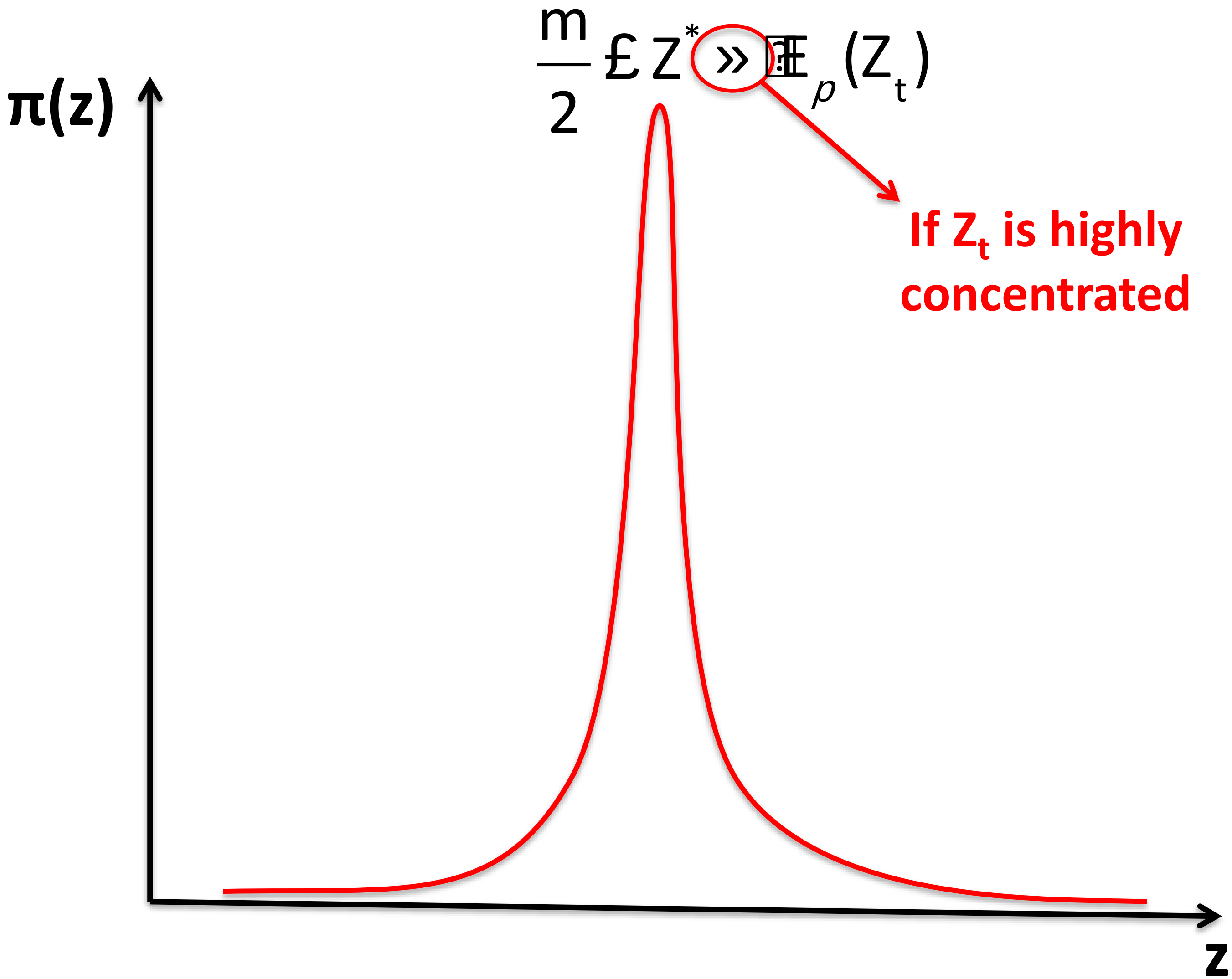
Patient Pool Size Balance Equation



≈(m/2)(1-$e^{-d/2}$) 2d/4m m/2 1/2

$$p(z-1)m = p(z+1)(z+1)\left(1-\left(1-\frac{d}{m}\right)^{z}\right) + ?p(z)z$$

Patient Pool Size Distribution

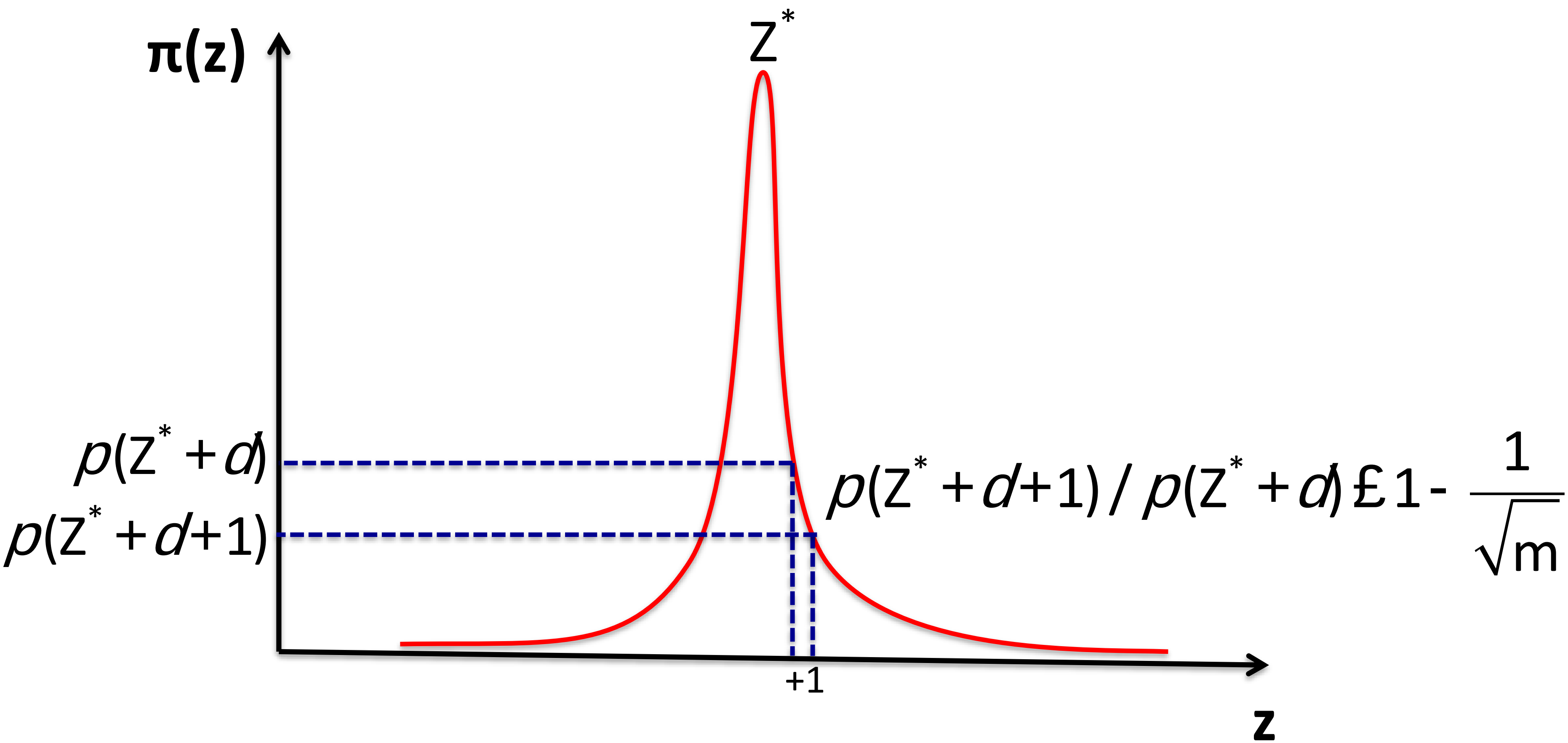


Patient Concentration Lemma

Lemma: For any ε > 0, there exist a Z^* > m/2 such that:

$$\Pr\left(Z^* - m^{\frac{1}{2}+e} < Z_t < Z^* + m^{\frac{1}{2}+e}\right) \xrightarrow{m\to\infty} 1$$

Patient Concentration Proof



$$\Rightarrow \pi(Z^*+\mu^{\frac{1}{2}+\varepsilon})/\pi(Z^*) \leq \left(1-\frac{1}{\sqrt{\mu}}\right)^{\mu^{\frac{1}{2}+\varepsilon}} \approx e^{-\mu^{\varepsilon}} \rightarrow 0$$ **QED!**

Planner's Trade-off

- Ideally, the planner aims to maximize:

The probability that a random agent has at least 1 edge

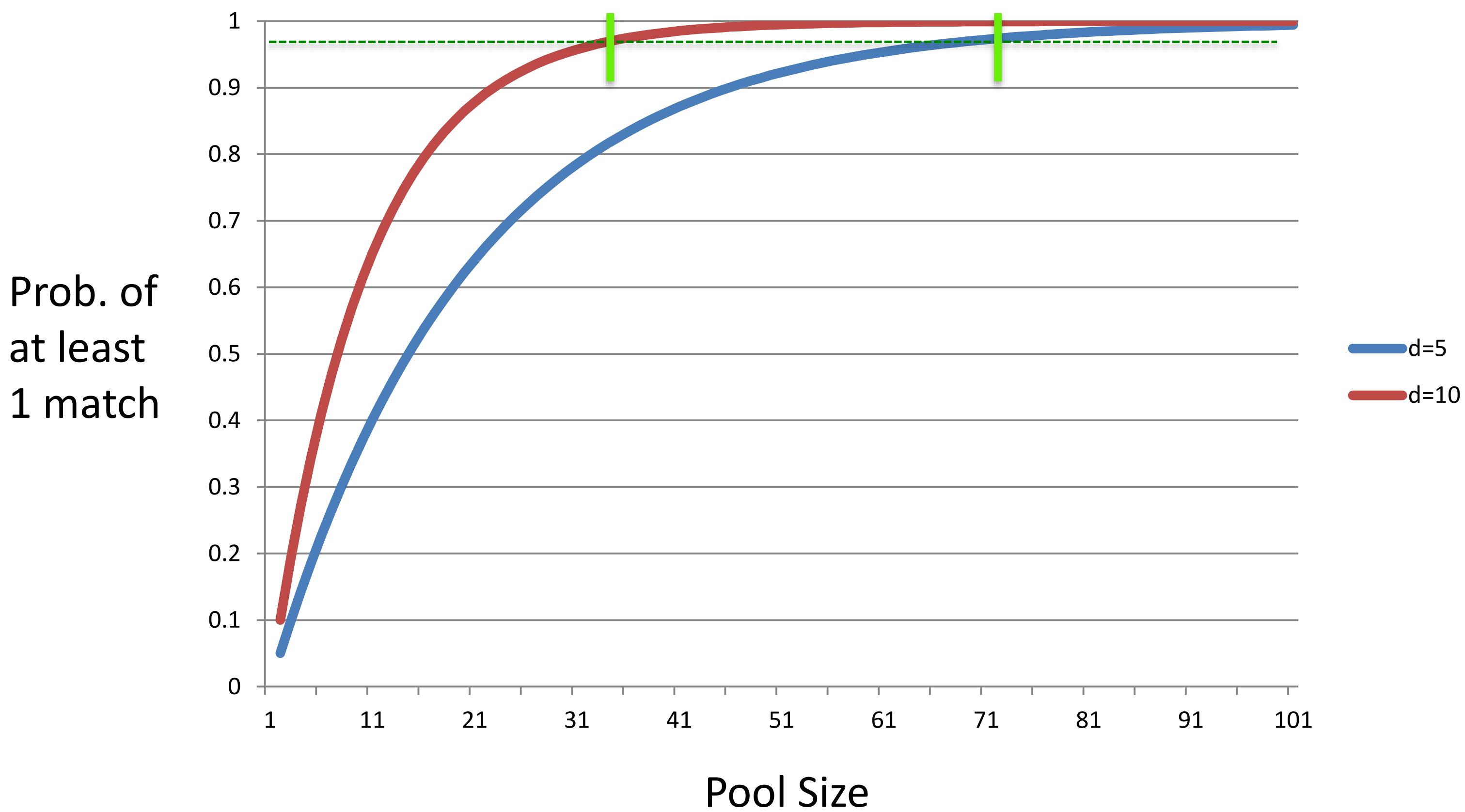
$$= 1 - \left(1 - \frac{d}{m}\right)^{\text{pool size}}$$

Market Thickness

- This is maximized by waiting and increasing pool size.

- But waiting is **costly**.

Increasing *d* and Market Thickness



Upper Bounding the Patient

First show that, $\tau_{mix}(\varepsilon) \leq O(\log(m) \cdot \log(1/\varepsilon))$.

Lemma: For any $T > 0$ and $\varepsilon > 0$,

$$\text{Loss(Patient)} \leq e^{-d/2} / 2 + \tau_{mix}(\varepsilon)/T + \varepsilon \cdot m / d^2$$

π: stationary distribution of the pool size

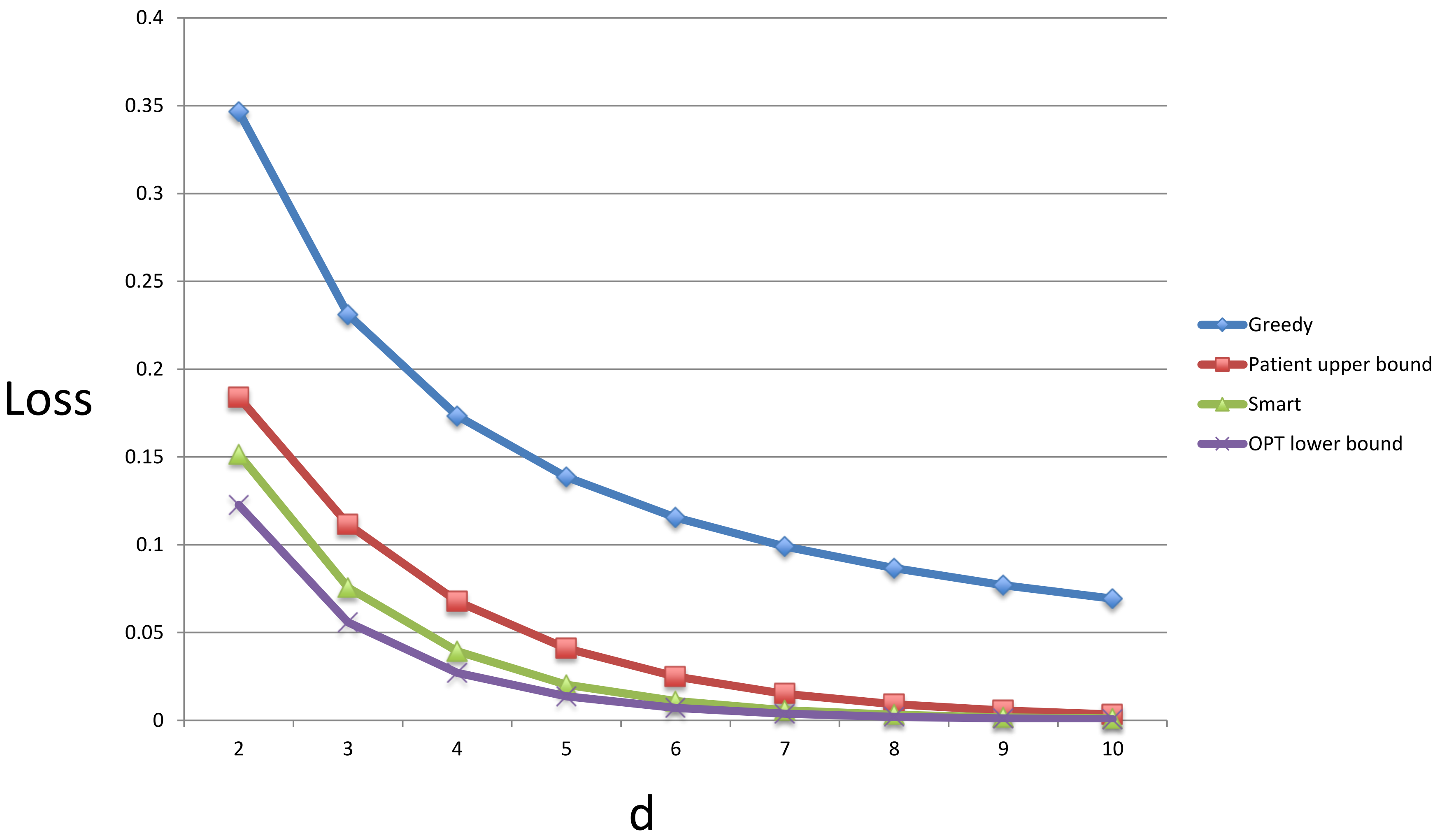
$E_\pi(Z_t)$: expected value of the pool size

Mixing Time and Total Variation

Definition. Let π be the stationary distribution of the Markov chain and $\mathbf{z_t}$ be its distribution at time *t*, then the *mixing time* of this chain is defined as:

$$\tau_{mix}(\epsilon) = \inf \{ t : \| z_t - \pi \|_{TV} := \sum_{k=0}^{\infty} | \pi(k) - z_t(k) | \le \epsilon \}$$

Smart Patient



Hazard Rate

	Time					
	6 mo.	1 yr.	2 yr.	3 yr.	4 yr.	5 yr.
On dialysis (for kidneys)	84%	75%	61%	50%	42%	34%