Prices and Auctions in Markets with Complex Constraints

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Product Heterogeneity

- What is a product? Debreu described commodities by their physical characteristics and their time & place of availability.
  - My return flight to SFO will use "time-space" resources.
  - Electricity in Palo Alto, CA at 2:00pm is not the same commodity as electricity in Palo Alto at 2:05pm, but in practice, the market price mechanism does not distinguish those.

- Debreu chapter 7 adds contingencies!

- When a product category lumps together heterogeneous items, there may be more constraints than just total resource constraints.

Violating Resource Constraints

- The cost of violating a resource constraint can be much higher than recognized by traditional neoclassical models
  - Electricity markets: brown-outs
  - Airlines: mid-air crashes

Beyond Resource Constraints

- Standard assumptions incorporated in neoclassical economic formulations
  - Static equilibrium: The only constraints on feasible allocations are “resource constraints”: demand must not exceed supply.
  - Equilibrium dynamics: The only losses incurred when demand exceeds supply is that some potential user is unserved.

- Failures of these two assumptions are a big part of the foundation of market design, requiring…
  1. matching within a product category
  2. additional constraints

Heterogeneity and Constraints

EXAMPLES

Front Range Spaceport
6 miles west of Denver Airport

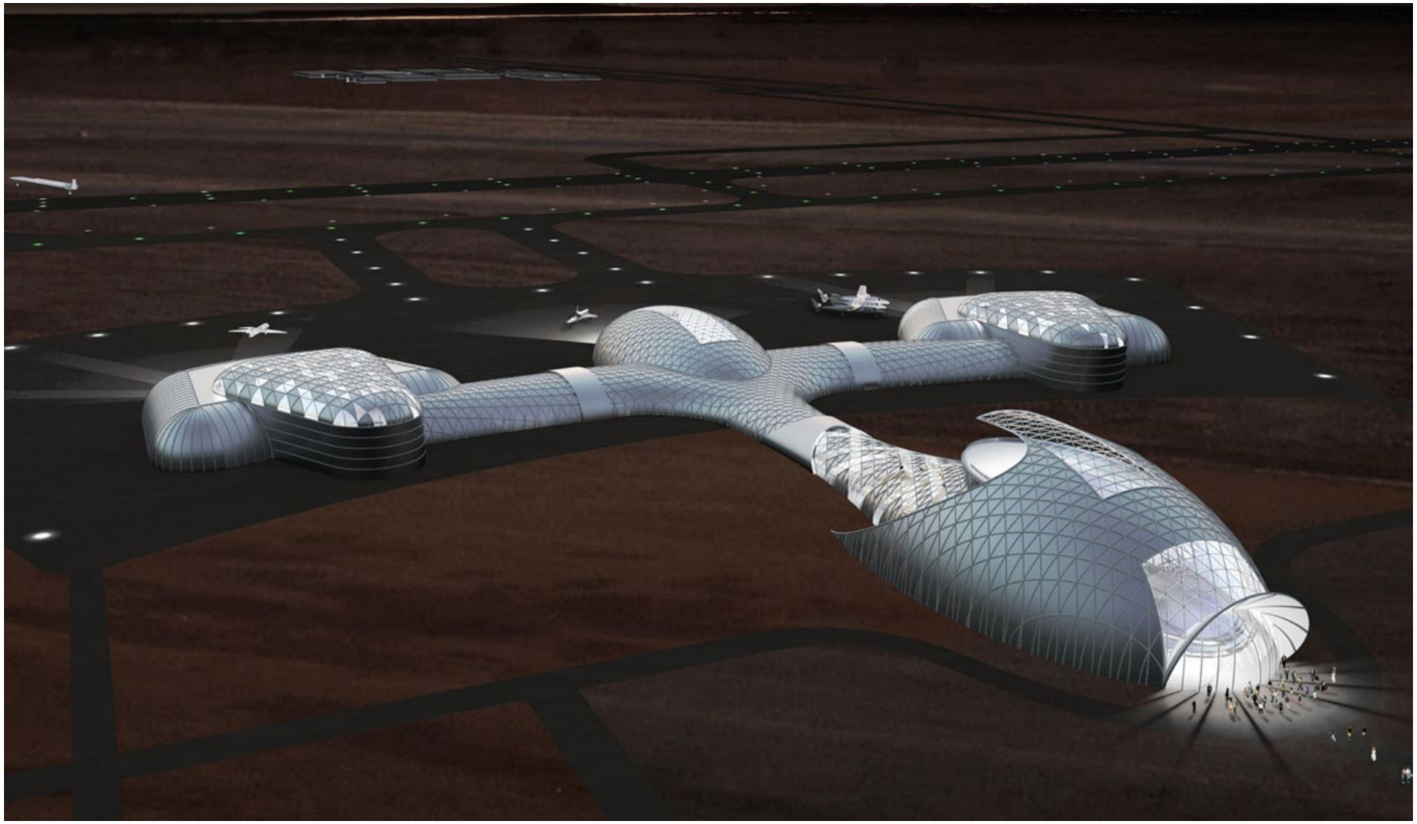


Illustration by Luis Vidal + Architects

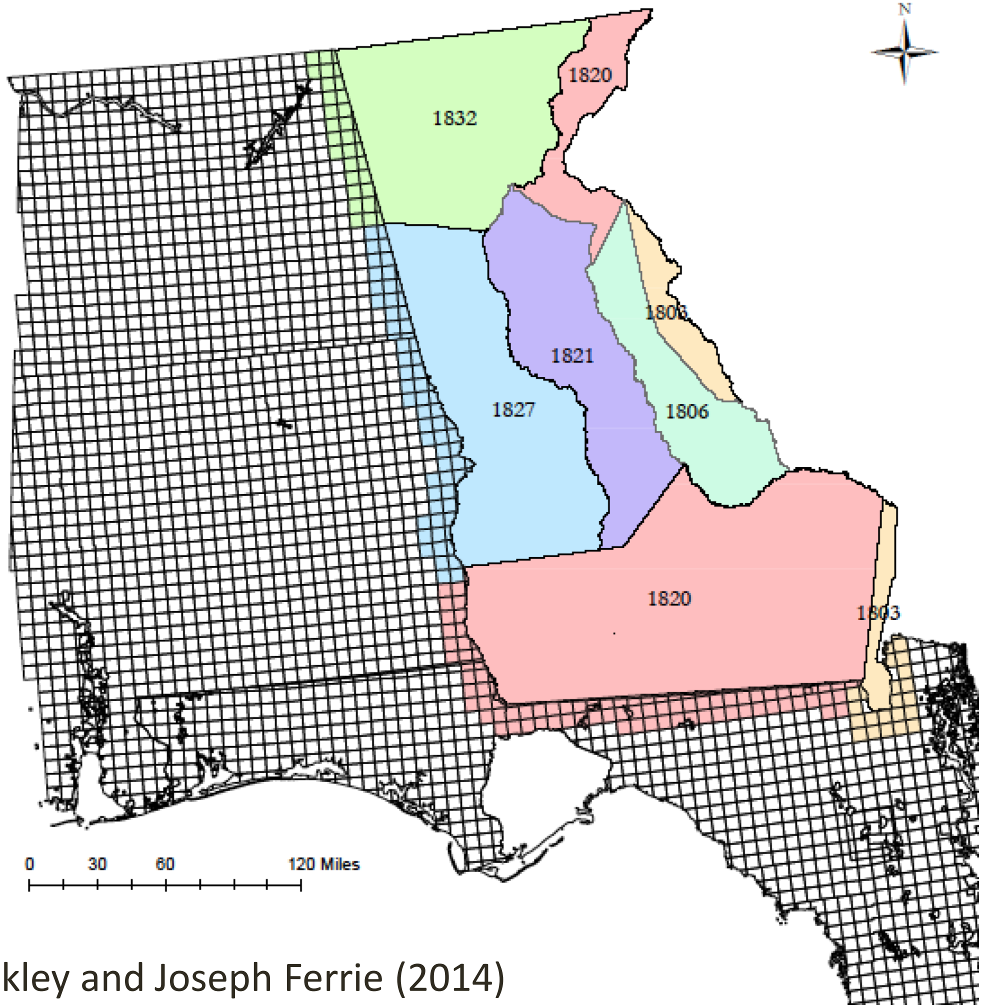
- Air traffic decisions are partly centralized, partly decentralized
- …but prices might help to guide better location and investment decisions.

Virgin Galactic

The VSS Enterprise in October 30, 2014

On October 31, 2014, Virgin Galactic's "VSS Enterprise" crashed spreading debris over 34 miles of the Mojave Desert.

Georgia Frontier Lands
Allocated 1803-1832



Hoyt Bleakley and Joseph Ferrie (2014)

Land Lotteries in Georgia

Hoyt Bleakley and Joseph Ferrie (2014)

Coase Theorem?

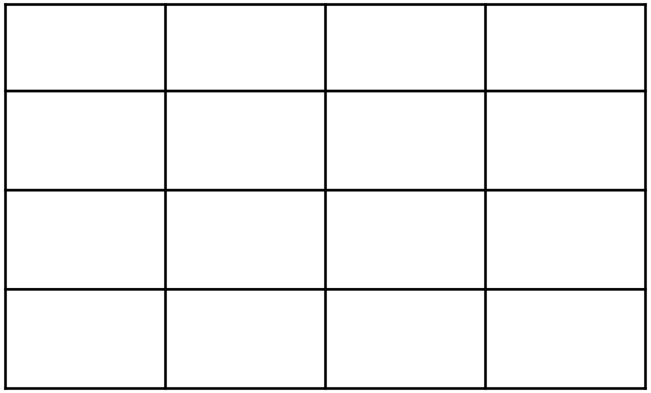
- Using tax records, Bleakley and Ferrie find that…
  - Initial Georgia land allocations changed little for 80+ years
  - Resulting in ~20% loss in land values
  - Allocations had become “unstuck” after about 150 years
- Mitigations using a designed market
  - Alternative property rights:
    - Example: users of property get options to buy nearby undeveloped properties
  - Multi-lateral transactions
    - Example: Developer buys multiple parts and subdivides.
  - Centralized procedures.
    - Example: a single large-scale auction event.
  - ***Variants of all of these are being employed for radio spectrum!***

Plots of Land are All Unique

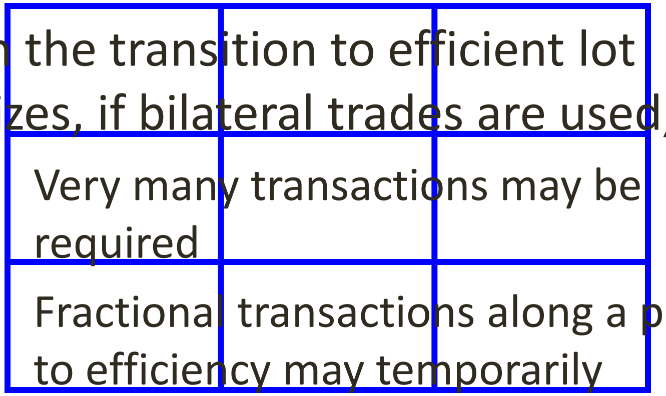
- Seattle, WA
- San Antonio, TX

The Reallocation Challenge

- The initially allocated plots of land were small.



- …but technical change led to larger optimal plots.

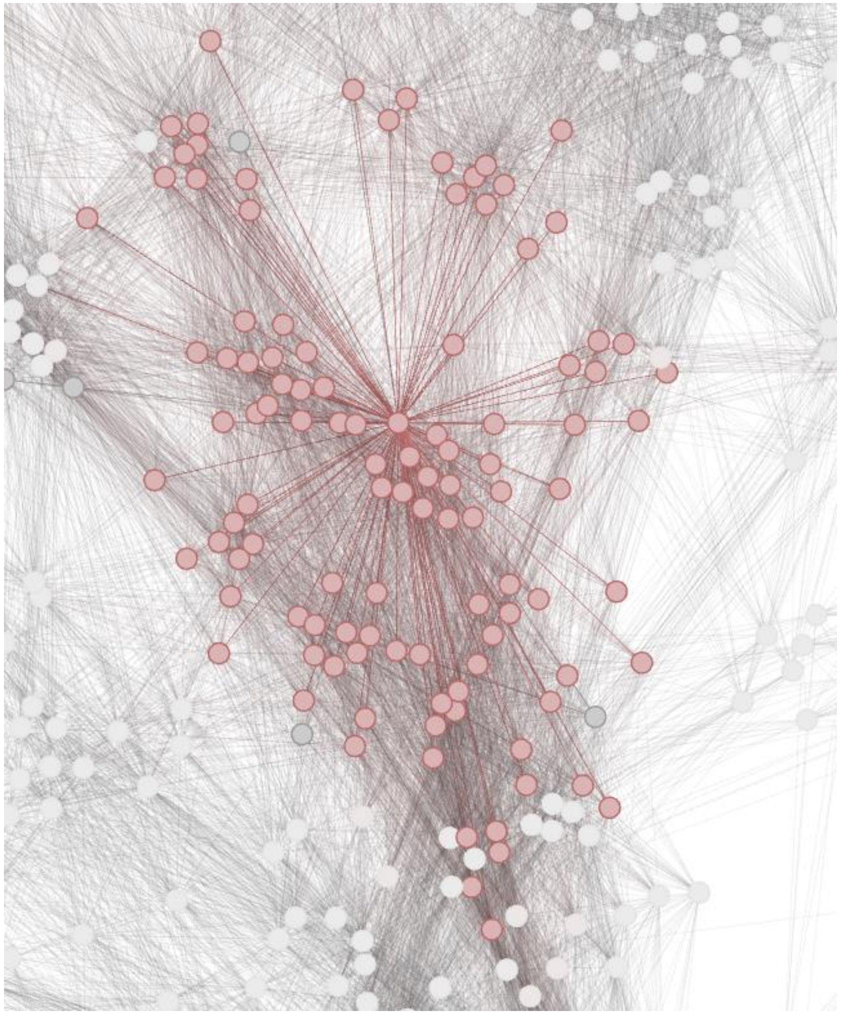


- In the transition to efficient lot sizes, if bilateral trades are used,
  - Very many transactions may be required
  - Fractional transactions along a path to efficiency may temporarily reduce some plot sizes and reduce total output.
  - Several individual plot owners may have hold-out power that could scuttle an efficient transition.

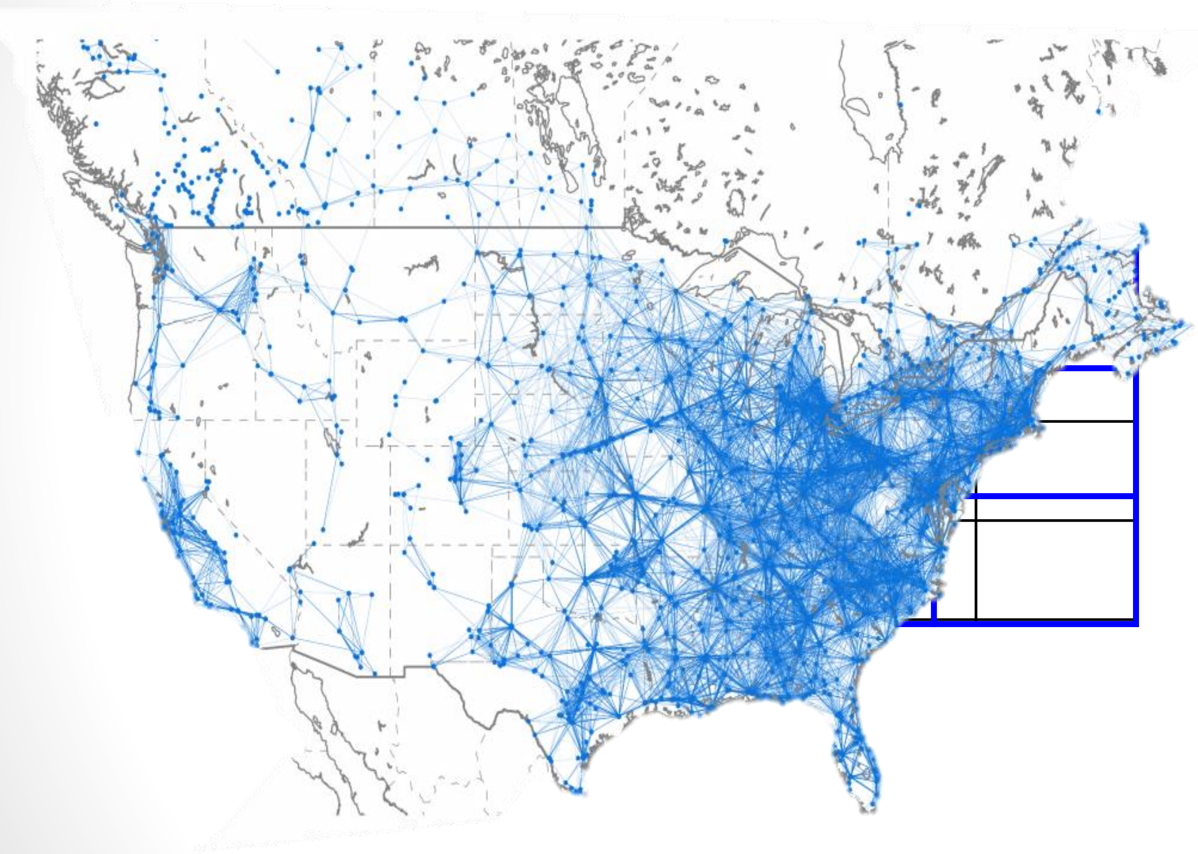
The Economic Setting

- TV broadcast
  - ~2200 UHF TV broadcasters in the United States + 800 in Canada
  - Currently using channels 14-36 and 38-51
  - 90% of viewers use cable or satellite (as of 2012)
  - Stations can share a digital channel by multiplexing
- Mobile broadband
  - Rapidly growing demand and value
  - Most useful low frequency spectrum is already allocated
- Plan
  - Transition some frequencies to higher valued uses
  - Provide a cash incentive for broadcasters to relinquish spectrum
  - A market will how many channels in the transition
  - Net positive revenue for the government

Co-Channel Interference Around One Station



From Reallocating Land…
To Reallocating Radio Spectrum



About 130,000 co-channel interference constraints, and about 2.7 million constraints in the full representation!

The graph-coloring is NP-complete. The FCC may sometimes be unable to determine, in reasonable time, whether a certain set of stations can be assigned to a given set of channels.

An “In-Between” Model of O’Hare Airport?!?

PRICES AND INCENTIVES IN THE “KNAPSACK PROBLEM”

Knapsack Problem

- Notation
  - Knapsack size: $\bar{S}$.
  - Items $n = 1, \ldots, N$
  - Each item has a value $v_n > 0$ and a size $s_n > 0$.
  - Inclusion decision: $x_n \in \{0,1\}$.

- Knapsack problem:

$$V^* = \max_{x \in \{0,1\}^N} \sum_{n=1}^{N} v_n x_n \text{ subject to } \sum_{n=1}^{N} s_n x_n \leq \bar{S}.$$

- The class of knapsack problems (verification) is NP-complete, but there are fast algorithms for “approximate optimization.”

Dantzig's "Greedy Algorithm"

- Order the items so that $\frac{v_1}{s_1} > \cdots > \frac{v_N}{s_N}$.

- Algorithm:
  1. $S_1 \leftarrow \bar{S}$. (Initialize "available space")
  2. For $n = 1, \dots, N$
  3. If $s_n \leq S_n$, set $x_n = 1$, else set $x_n = 0$.
  4. Set $S_{n+1} = S_n - x_n s_n$.
  5. Next n.
  6. End

- The items selected are $\alpha^{Greedy}(v; s) \stackrel{\text{def}}{=} \{n | x_n = 1\}$.
  - This function is "monotonic."

"Nearly the Same" Algorithm Can be Formulated as an Auction

- Let $p(0) > \frac{v_1}{s_1}$; $S(0) = \bar{S}$; and label all n "Out."

Discrete greedy algorithm

- For $t = 1,2, \dots p(0)/\varepsilon$.
  - Mark "Rejected" any n for whom $s_n + S(t-1) > \bar{S}$
  - Set $p(t) = p(t-1) - \varepsilon$ (where $\varepsilon > 0$ is the "bid decrement")
  - If some n who is marked "Out" has $v_n > p(t)s_n$, mark it "Accepted" and set $S(t) = S(t-1) - s_n$.
- Next t

Generalizable Insight:

- Close equivalence between various clock auctions and related greedy algorithms (Milgrom & Segal).

The LOS Auction

- **Model** (*Lehman, O'Callaghan and Shoham (2002)*)
  - Each item *n* is owned by a separate bidder.
  - Sizes s_n are observable to the auctioneer.
- **"LOS" Direct Mechanism**
  - Each bidder n reports its value v_n.
  - Allocate space to the set of bidders $\alpha^{Greedy}(v; s)$.
  - Charge each bidder its "threshold price", defined by:

$$p_n(v_{-n}; s) \stackrel{\text{def}}{=} \inf\{v'_n \middle| n \in \alpha^{Greedy}(v'_n, v_{-n}; s)\}.$$

- ***Theorem***. *The LOS auction is truthful.*

LOS Mechanism Can Lead to Excessive Investments

- Consider a game in which, before the auction, each bidder n can, by investing c, reduce the size of its item to $s_n - \Delta$.

- **Examples:** Suppose there are N items, each of size 1 and the knapsack has size $N - 1$, so $\alpha^{Greedy}(v, s) = \{1, \ldots, N - 1\}$.
- **Losses from excessive investment:** If $\Delta = \frac{1}{N}$ and $c = v_N - \varepsilon$, then there is a Nash equilibrium in which all invest, even though the cost is nearly *N* times the benefit.

- *Can a uniform price mechanism perform similarly well?*

Uniform-Price Greedy Mechanism

- **Determine the allocation**

1. Order the items so that $\frac{v_1}{s_1} > \cdots > \frac{v_N}{s_N}$.
2. Initialize $n \leftarrow 1$ and set $S_1 \leftarrow S$.
3. If $s_n > S_n$, go to step 5
4. Increment *n* and go to step 3
5. Set $\alpha^{Alt}(v, s) \leftarrow \{1, \ldots, n-1\}$.

- **Set a supporting price (a "uniform" price of space)**

6. Define $\hat{p} \overset{\text{def}}{=} v_n/s_n$ (the "pseudo-equilibrium" price of space)
7. Set $p_j \leftarrow \hat{p}s_j$ for $j = 1, \ldots, n-1$, and $p_j \leftarrow 0$ for $j = n, \ldots, N$.
8. End

- Define $V^{Alt}(v, s) \overset{\text{def}}{=} \sum_{n \in \alpha^{Alt}(v,s)} v_n$.

Packing Efficiency and Truthfulness

- **Theorem**. The "Alt" mechanism is truthful. Moreover,

$$V^{Greedy}(v,s) \geq V^{Alt}(v,s) \geq V^{*}(v,s) - \left(S - S^{Alt}\right)\hat{p}.$$

  - The bound on efficiency loss is observable: it is equal to the "value" of the unused space in the knapsack.

Investment Efficiency

- Suppose that each bidder *n* can, by expending $c_n \in C$, determine the size $s_n(c_n)$ of its item, where C is a finite set. Let n^* denote the index of the first bidder not packed by the Alt mechanism.
- Notation:

$$V^{**} \stackrel{\text{def}}{=} \max_{\{x,c | c_{n^*} = \cdots = c_N = 0\}} \sum_n (x_n v_n - c_n) \ s.t. \sum_n x_n s_n(c_n) \le S$$

$$c_n^* \stackrel{\text{def}}{=} \underset{c_n \in C}{\operatorname{argmax}} \max_{x_n} v_n x_n - \hat{p} s_n(c_n) - c_n$$

- **Theorem**. The combined loss from packing and investment in *Alt* is ("observably") bounded as follows:

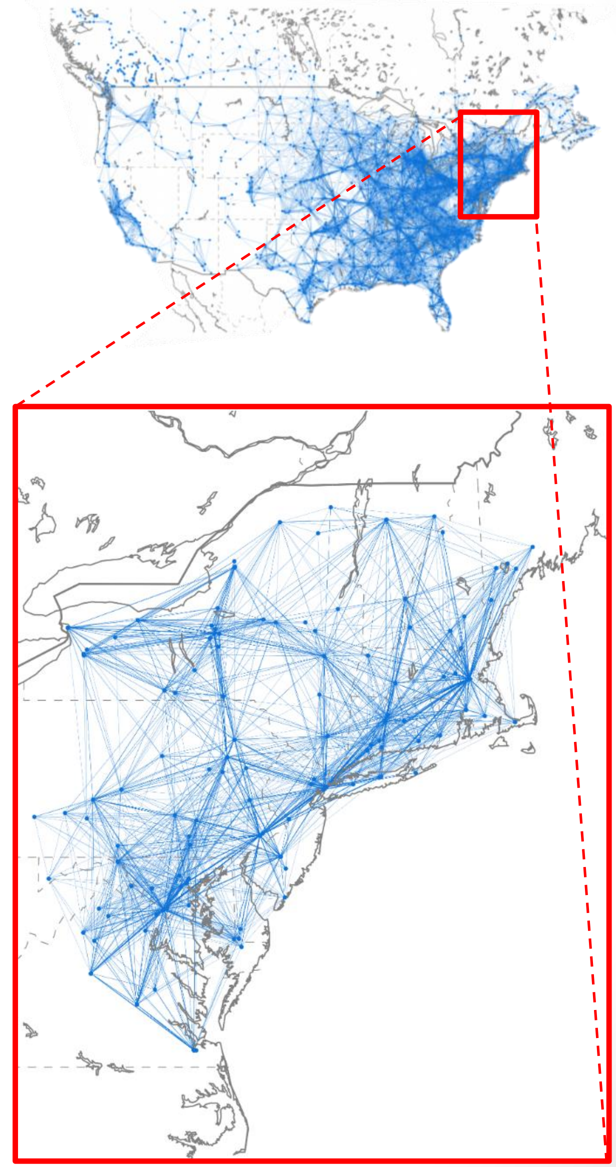
$$V^{**} - V^{Alt}(s(c^*)) \le \left(S - S^{Alt}(c^*)\right) \hat{p}$$

Greedy Algorithms and Auctions

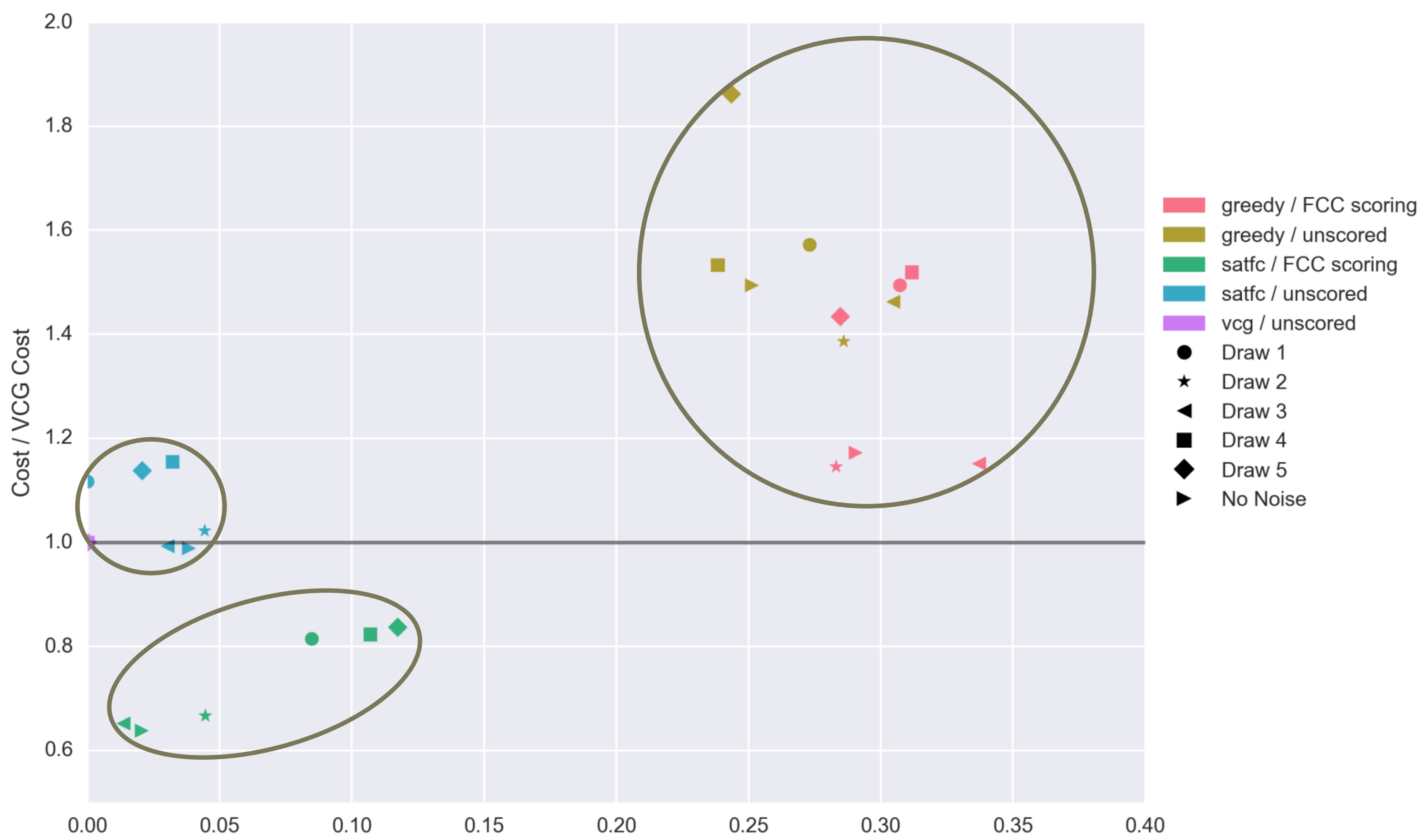
- In an auction with single-minded bidders, a greedy algorithm can be used to sort the bidders into two sets, winners and losers.
  - LOS Algorithm and Related: Select winners by a greedy algorithm; other bidders are losers.
  - DAA Algorithm and Related: Select losers by a greedy algorithm; other bidders are winners.
- The two categories are *economically* distinct because
  - winners collect payments
  - losers do not
- The FCC descending clock auction is a DAA algorithm.
  - One could specify an ascending clock for an LOS-style algorithm.

How Well Does The Incentive Auction Work?

- Can’t run VCG on national-scale problems: can’t find an optimal packing
  - Restrict attention to all stations within **two constraints of New York City**
    - a very densely connected region
    - 218 stations met this criterion
- Reverse auction simulator (UHF only)
- Simulation **assumptions**:
  - 100% participation
  - 126 MHz clearing target
  - valuations generated by sampling from a prominent model due FCC chief economist (before her FCC appointment)
  - 1 min timeout given to SATFC



Comparative Performance of Incentive Auction Algorithms in Simulations



$$Loss = \frac{Value\ of\ Stations\ on\ Air}{VCG\ Value} - 1$$

More Greedy Algorithms

GREEDY ON MATROIDS

Matroid Terms Defined

- Given a fine “ground set” X, let $\wp(X)$ denote its power set.
  - Example: X is the set of rows of a finite matrix
- Let $\mathcal{R} \subseteq \wp(X)$ be non-empty and all its elements “independent sets.”
  - Example: all linearly independent sets of rows in X.
- A “basis” is a maximal independent set.
  - Example: a maximal linearly independent set of rows of X.

- The pair $(X, \mathcal{R})$ (or just the set $\mathcal{R}$) is a *matroid* if
  1. [Free disposal] If $S' \subseteq S \in \mathcal{R}$, then $S' \in \mathcal{R}$.
  2. [Augmentation Property] Given $S, S' \in \mathcal{R}$, if $|S| > |S'|$, then there exists $n \in S - S'$ such that $S' \cup \{n\} \in \mathcal{R}$.

Greedy Algorithm

- Given any collection of independent sets $\mathcal{R}$.
- Order the items so that $v_1 > \cdots > v_N$. (No volumes)
- Algorithm:
  1. Initialize $S_0 \leftarrow \emptyset$.
  2. For $n = 1, \dots, N$
  3. $S_n \leftarrow \begin{cases} S_{n-1} \cup \{n\} \text{ if } S_{n-1} \cup \{n\} \in \mathcal{R} \\ S_{n-1} \quad \text{otherwise} \end{cases}$
  4. Next n
  5. Output S_N.

Optimization on Matroids

- For simplicity, assume a unique optimum.
- **Theorem.** If $\mathcal{R}$ is a matroid and S_N is the greedy solution, then

$$S_N = \underset{S \in \mathcal{R}}{\operatorname{argmax}} \sum_{n \in S} v_n$$

- **Intuition**. Suppose that $S^* = \{i_1, \dots, i_k\} \in \mathcal{R}$ does not include the most valuable item, which is item 1.
  - Then S^* is not optimal, because we can *augment* the set $\{1\}$ using items from S^* to create a *k* item set that is strictly more valuable.
  - *Notice that this means that if we have to choose between two items, then the greedy has identical continuations after both.*

Full Proof is by Induction

- Suppose that the set selected by the greedy algorithm is $\{g_1, \dots g_k\}$ and that the g_n is the element with the lowest index that such that for the optimal set S, $g_n \notin S$. So, $S = \{g_1, \dots, g_{n-1}\} \cup S'$ and for each element $s \in S'$, $v_{g_n} > v_s$.
- By the augmentation property, it is possible to augment $\{g_1, \dots, g_n\}$ to a basis B set by iteratively adding elements from S', while omitting just one element, say $\hat{s}$.
- By then S was not optimal, because B is better:

$$\sum_{j \in B} v_j - \sum_{j \in S} v_j = v_{g_n} - v_{\hat{s}} > 0. \quad \blacksquare$$

Matroids and Substitutes

- Let $\mathcal{R}$ be a non-empty collection of independent sets satisfying free disposal.
- For each good in $x \in X$, there is a buyer $v(x)$ and a price $p(x)$. The buyer's demand is described by:

$$V^*(\mathcal{R}, v) \stackrel{\text{def}}{=} \max_{S \in \Re} \sum\nolimits_{x \in S} (v(x) - p(x))$$

$$d^*(p|\mathcal{R}, v) \stackrel{\text{def}}{=} \operatorname*{argmax}_{S \in \Re} \sum\nolimits_{x \in S} (v(x) - p(x))$$

- **Theorem**. The items in X are substitutes if and only if $\Re$ is a ***matroid***.
- **Intuition**: Raise the price of item x. When it becomes too expensive and is "replaced by" some item y, the items chosen before and after in the greedy algorithm are unaffected.

Proof Sketch

- Suppose that $n \in d^*(p|\mathcal{R}, v)$ and consider a price $p'(n) > p(n)$ such that $n \notin d^*(p\backslash p'(n)|\mathcal{R}, v)$. Let $n' \notin d^*(p|\mathcal{R}, v)$ be the first new item chosen instead during the greedy algorithm with prices $p\backslash p'(n)$. Let the state of the greedy algorithm when it is chosen be S' and let $S = S' \cup \{n\} - \{n'\}$.
- By the augmentation property, the the feasible next choices to augment S' and S are identical. Hence, $d(p\backslash p'(n)|\mathcal{R}, v)$ $= (d(p|\mathcal{R}, v) - \{n\}) \cup \{n'\}$, as required.

- Conversely, if $\mathcal{R}$ is not a matroid, then…

Necessity of Matroids

- **Theorem.** If $\mathcal{R}$ is a non-empty family that satisfies free disposal but not the augmentation property, then there is some vector of values v such that (the greedy algorithm "fails") $S_N \notin \underset{S \in \mathcal{R}}{\operatorname{argmax}} \sum_{n \in S} v_n$.

- **Proof**. $\mathcal{R}$ does not have the augmentation property, so there is some $S, S' \in \mathcal{R}$ such that $|S| > |S'|$ and there is no $n \in S - S'$ such that $S' \cup \{n\} \in \mathcal{R}$.
- Let $\epsilon > 0$ be small and take:

$$v_n = \begin{cases} 1 & \text{if } n \in S' \\ 1 - \epsilon & \text{if } n \in S - S' \\ 0 & \text{otherwise} \end{cases}$$

- Then the greedy algorithm selects S' and no elements of $S - S'$, so its value is $|S'|$, but S achieves at least $(1 - \epsilon)|S| > |S'|$. ■

Approximate Substitutes and the Incentive Auction

WHY SHOULD WE CARE?

The Substitution Index

- Why does the DA algorithm perform so well?
- Two conjectured reasons:
  - Special *constraints*: the independent sets $\mathcal{C}$?
  - Special *values*: a set $\mathcal{O} \subseteq C$ where the optimum may lie?
    - “Zero knowledge case”: $\mathcal{O} = C$.

- **Definitions**. Given the ground set $\mathcal{X}$ and the constraints $\mathcal{C}$ and possible optimizers $\mathcal{O}$ that both satisfy free disposal,

$$\mathcal{R}^*(\mathcal{C},\mathcal{O}) \stackrel{\text{def}}{=} \underset{\substack{\mathcal{R}\ a\ matroid \\ \mathcal{R}\subseteq\mathcal{C}}}{\operatorname{argmax}} \min_{X\in\mathcal{O}} \max_{\substack{X'\in\mathcal{R} \\ X'\subseteq X}} \frac{|X'|}{|X|}$$

$$\rho(\mathcal{C},\mathcal{O}) \stackrel{\text{def}}{=} \max_{\substack{\mathcal{R}\ a\ matroid \\ \mathcal{R}\subseteq\mathcal{C}}} \min_{X\in\mathcal{O}} \max_{\substack{X'\in\mathcal{R} \\ X'\subseteq X}} \frac{|X'|}{|X|}$$

Approximation Theorem

- Given the ground set $\mathcal{X}$, any $\mathcal{S} \subseteq \mathcal{P}(\mathcal{X})$ and any $v \in \mathbb{R}_+^{\mathcal{X}}$, define notation as follows:

$$V^*(\mathcal{S}; v) \stackrel{\text{def}}{=} \max_{X \in \mathcal{S}} \sum_{n \in X} v_n$$

- **Theorem.** The greedy solution on $\mathcal{R}^*$ approximates the optimum in worst case as follows:

$$\min_{v > 0} \frac{V^*(\mathcal{R}^*; v)}{V^*(\mathcal{O}; v)} = \rho(\mathcal{C}, \mathcal{O}).$$

Proof Sketch, 1

- Let
$$v^* \in \operatorname*{argmin}_{v>0} \frac{V^*(\mathcal{R}^*; v)}{V^*(\mathcal{O}; v)}, \rho^* = \frac{V^*(\mathcal{R}^*; v^*)}{V^*(\mathcal{O}; v^*)}$$
- Among optimal solutions, choose v^* to be one with the smallest number of strictly positive components.
- Without loss of optimality, we rescale v^* so that the smallest strictly positive component is 1.
- The next step will show that every component of v^* is zero or one, so that the values of the two minimization problems must exactly coincide.

Proof Sketch, 2

- Consider the family of potential minimizers $\hat{v}(\alpha)$, where

$$\hat{v}_n(\alpha) \stackrel{\text{def}}{=} \begin{cases} \alpha & \text{if } v_n^* = 1 \\ v_n^* & \text{otherwise} \end{cases}$$

- Then, $v^* = \hat{v}(1)$. The value of the objective for $\hat{v}(\alpha)$ is

$$\hat{\rho}(\alpha) = \frac{V^*\big(\mathcal{R}^*; \hat{v}(\alpha)\big)}{V^*\big(\mathcal{O}; \hat{v}(\alpha)\big)} = \frac{\alpha\big|\hat{X} \cap X_{\mathcal{R}^*}\big| + \sum_{n\in(\mathcal{X}-\hat{X})\cap X_{\mathcal{R}^*}} v_n^*}{\alpha\big|\hat{X} \cap X_{\mathcal{O}}\big| + \sum_{n\in(\mathcal{X}-\hat{X})\cap X_{\mathcal{O}}} v_n^*}$$

where

$$\hat{X} \stackrel{\text{def}}{=} \{n | v_n^* = 1\}$$

$$X_{\mathcal{O}} \in \underset{S\in\mathcal{O}}{\operatorname{argmax}} \sum_{n\in S} v_n^*$$

$$X_{\mathcal{R}^*} \in \underset{S\in\mathcal{R}^*}{\operatorname{argmax}} \sum_{n\in S} v_n^*$$

Proof Sketch, 3

$$\hat{\rho}(\alpha) = \frac{\alpha\left|\hat{X} \cap X_{\mathcal{R}^*}\right| + \sum_{n \in (\mathcal{X} - \hat{X}) \cap X_{\mathcal{R}^*}} v_n^*}{\alpha\left|\hat{X} \cap X_{\mathcal{P}}\right| + \sum_{n \in (\mathcal{X} - \hat{X}) \cap X_{\mathcal{O}}} v_n^*}$$

For $\hat{\rho}(\cdot)$ to achieve its minimum of ρ^* when $\alpha = 1$, it must be a constant function, which requires $\frac{|\hat{X} \cap X_{\mathcal{R}^*}|}{|\hat{X} \cap X_{\mathcal{P}}|} = \rho^*$. Then, since v^* is the minimizer with the fewest strictly positive elements, $\{n | v_n^* > 1\} = \emptyset$. ■

The US Incentive Auction

WHAT IS GOING ON NOW?

Current Status

Incentive Auction Dashboard - Stage 2

Bidding in the clock phase of the reverse auction will begin September 13, 2016.

Clearing Target	114 MHz
Licensed Spectrum	90 MHz

Final Stage Rule

1 First Component

Auction Proceeds

Target	Actual
$15,896,290,987	$23,108,037,900

2 Second Component

Net Proceeds

Target	Estimated
$88,379,558,704	$22,450,000,000

Final Stage Rule

Not Met

As of Stage 1

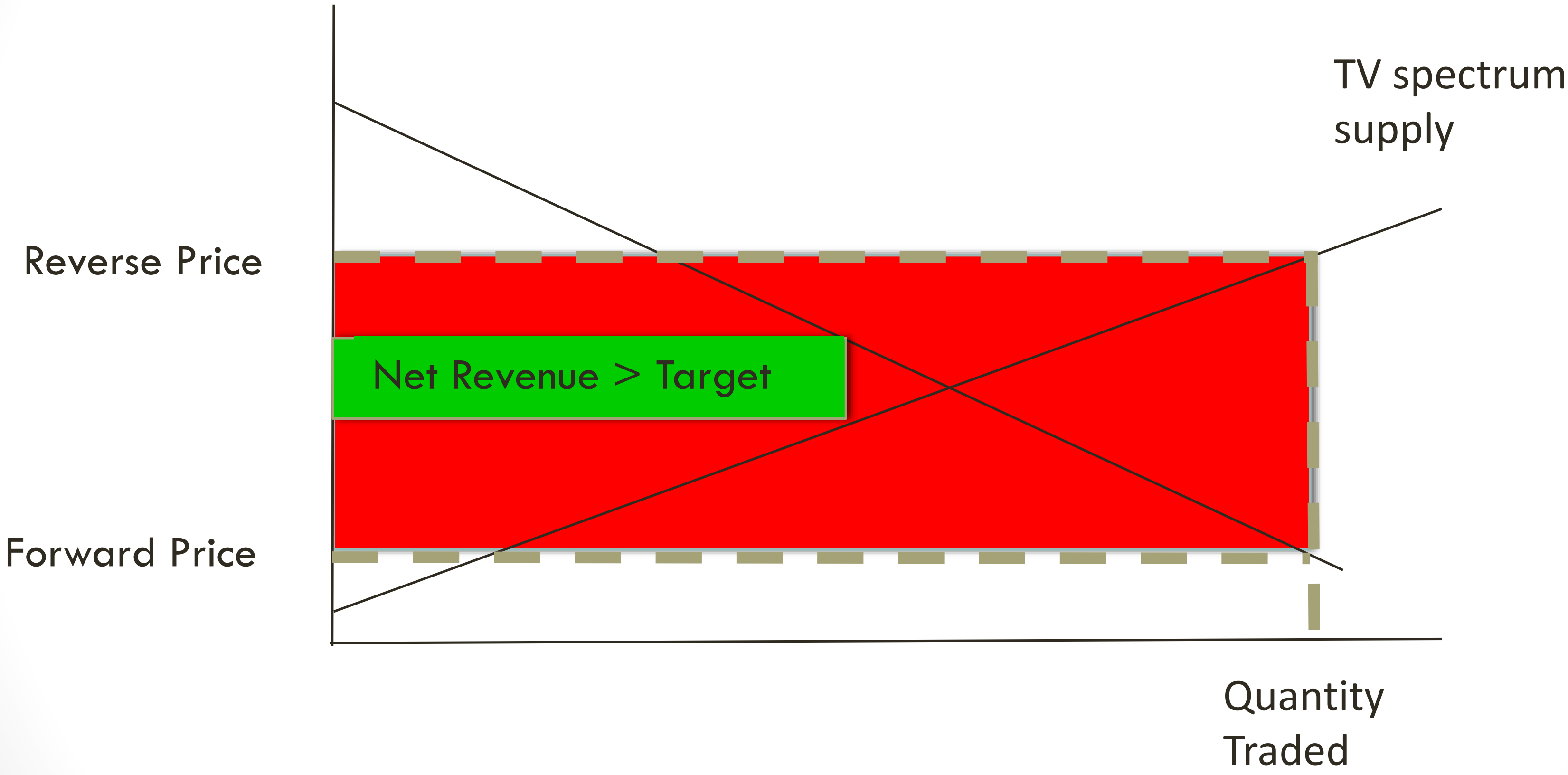
Reverse Auction

Current Round	Bidding not started
Clearing Cost	N/A

Forward Auction

Current Round	Stage 1 concluded
Auction Proceeds as of Stage 1	$23,108,037,900

The Incentive Auction “Stages”: A Conceptual Illustration



Thank you!

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